Position Auctions with Budget-Constrained Advertisers

First version: May 20, 2012  
This version: July 22, 2014

Shijie Lu  
Doctoral Candidate of Marketing  
Marshall School of Business  
University of Southern California  
shijielu@usc.edu

Yi Zhu  
Assistant Professor of Marketing  
Carlson School of Management  
University of Minnesota, Twin Cities  
yizhu@umn.edu

Anthony Dukes  
Associate Professor of Marketing  
Marshall School of Business  
University of Southern California  
dukes@marshall.usc.edu

Acknowledgement: We thank for extensive comments and discussions with Vibhanshu Abhishek, Tony Cui, Liang Guo, Jidong Han, Kristiaan Helsen, Peter Popkowski Leszczyc, De Liu, Amin Sayedi, Matthew Selove, Jeffrey Shulman, Wenbo Wang, Ken Wilbur, Linli Xu, Zelin Zhang, Ying Zhao and seminar participants at the 2012 INFORMS Marketing Science Conference, USC-Marketing Research Fair, the INFORMS Annual Meeting 2012, Hong Kong University of Science and Technology and Renmin University. The second author gratefully acknowledges financial support from the Dean’s Small Research Grant at Carlson School of Management and 3M Nontenured Faculty Grant.
Abstract

This paper examines position auctions with budget-constrained advertisers, a dominant bidding environment used by publishers to allocate positions in online advertising. As we show, budget constraints play a crucial role in bidding strategy. We provide a comprehensive analysis of this auction and identify three new categories of bid strategies used by advertisers in equilibrium (i) aggressive: bid jamming; (ii) defensive: jamming protection; and (iii) semi-aggressive: budget pegging. We then examine the implication of budget sizes on advertisers’ profits and the publisher’s revenues. There exists a situation in which an advertiser’s profit strictly decreases with her budget. In addition, the publisher’s revenue can decrease when an advertiser’s budget increases. This happens whenever a budget increase (i) reduces the aggressiveness of bidding strategies or (ii) induces the high-value advertiser to bid for a higher position. In an extension, we consider advertisers’ budget decisions being endogenous and discover an inverted-U relationship between the publisher’s revenue and the cost of expanding an advertiser’s budget. Several managerial implications for both advertisers and publishers are discussed.

Keywords: Position Auctions, Budget Constraints, Internet Marketing, Advertising, Game Theory.
1. Introduction

The position auction, typically a generalized second-price (GSP) auction, is the dominant format used by online publishers such as search engines (e.g., Google, Yahoo, and Bing), social media (e.g., Facebook and LinkedIn), and media websites (e.g., CNN.com and Fox.com) to allocate advertising space appearing in ranked listings. Advertisers are ranked in the ad listing according to their bids, with the highest bids receiving the best ranks, and pay a cost-per-click (CPC), which is determined by the bid of subsequently ranked advertiser. An advertiser’s total online advertising cost, however, depends not just on the bids in the auction, but also on the volume of clicks, or click-through-rate (CTR), made throughout the duration of the search listing.

In these online bidding environments, which can generate thousands of clicks on high-traffic publishers, marketers can inadvertently spend more than they budgeted for online advertising. To help marketers control their ad spending, publishers require every advertiser to indicate their daily budgets for each auction. This budget is the maximum amount of money a firm is able to spend in one day on a given position auction. This paper studies strategic bidding in position auctions with such budget constraints.  

Despite the recent advances in the theory of position auctions (Edelman et al. 2007 and Varian 2007), little is known about the impact of budget constraints on bidding behaviors. Furthermore, the existence of budget constraints can have significant effects on bidding strategies in position auctions. Whenever an advertiser is removed because its budget is exhausted, the next ranked advertiser rises up in position and enjoys improved CTR at her original CPC. This generates the possibility that an advertiser could strategically bid more aggressively in order to raise the cost of the advertiser ranked above her and subsequently move into a higher slot later. These strategic considerations have taken on

---

1 We assume that budgets submitted to the publisher are actual financial constraints. This simplification allows us to abstract away any strategic gaming by budget misreporting in order to focus on advertisers’ bid strategies.
additional relevance in light of the increased availability of competitive intelligence online. Major publishers such as Google provide free tools to allow advertisers to infer CPC across different positions. By linking the competitors’ observed positions with the CPC estimates, advertisers can conveniently acquire knowledge on competitors’ value-per-click. Moreover, third-party services like Spyfu, iSpionage, and AdGooroo keep track of advertising performance data from all major online publishers. Advertisers can utilize these services to learn the daily budgets of their rival advertisers and incorporate this knowledge when bidding for a keyword.

The importance of budgets for strategic bidding is also reflected in the field data we collected. These data indicate that advertisers with higher budgets tend to rank in higher positions. Since large-budget advertisers are not necessarily the bidders with the highest value-per-click, it suggests that budgets play a role in bidding behaviors and position outcomes.

The objective of this research is to understand bidding incentives in position auctions with budget constraints. We address the following questions: How and when should an advertiser’s bid depend on the size of its own budget? On the size of its rival’s budget? In addressing these questions, our research helps advertisers manage this strategic environment. We show that an advertiser’s optimal bid can depend on her own budget and the budget of her rival. This result raises a second set of questions: How does the size of an advertiser’s budget affect (i) its position; (ii) its profit; and (iii) the publisher’s revenue? As we show, larger-budget advertisers do not necessarily have more profit nor generate the most revenue for the publisher.

We build and study a game-theoretic model of bidders competing in an online position auction with limited budgets. The game comprises two budget-constrained advertisers competing for the top two ad positions sold by a publisher. Advertisers differ in their value-per-click and in their daily budgets. The level of competitive intensity (e.g., popularity of a keyword) is reflected by a reservation value, or

---

2 See Appendix A1 for more details.
the highest bid among a set of non-strategic, lower-ranked advertisers. The highest bidding advertiser (whom we call the “first advertiser”) attains the first position and pays a CPC equal to the second advertiser’s bid. The second advertiser’s CPC is the reservation value. If the first advertiser’s budget is exhausted before the end of the day, her advertisement is then removed and the second advertiser ascends to the first position, while continuing to pay the reservation value. The second advertiser retains the first position until the end of the day or until her own budget is exhausted. This simple game structure captures the bidding environment of the position auctions run by many online publishers, like those mentioned above. An equilibrium analysis of this game yields three general results.

First, equilibrium bidding in position auctions with budget constraints is dramatically different from previous results. In addition to the budget-free bid found in a setting when advertisers have sufficient budgets, we identify three new classes of bid strategies that advertisers may use in equilibrium. We classify these strategies by their level of aggressiveness: (i) bid jamming; (ii) jamming protection; and (iii) budget pegging. Each strategy is contingent on the advertisers’ budgets and values. Bid jamming is an aggressive bid strategy used by the lower-ranked advertiser. This advertiser targets her bid on the bid of the higher-ranked advertiser, bidding pennies below the bid in order to quickly exhaust the advertiser’s relatively small daily budget and ascend to the top position while maintaining its relatively low CPC. Jamming protection is a defensive bid strategy utilized by a small-budget bidder, bidding below the budget-free bid in the hope of discouraging the competitor’s use of bid jamming. This conservative strategy ensures that the advertiser’s profit is immune to the potential attack of bid jamming by the large-budget competitor. Finally, budget-pegging is a semi-aggressive bid strategy by an advertiser whose aim is to exhaust the budget of the advertiser in the higher position, but not too quickly. This happens when both advertisers’ budgets are small. Unlike bid-jamming, in which the second advertiser’s bid is a function of the first advertiser’s bid, in a budget-pegging strategy the bid is a
function of the first advertiser’s budget. The bid is “pegged” in a fixed proportion of the higher ranked bidder’s budget in such a way so as to slowly exhaust the first advertiser’s budget, ascend to the first position, and endure the remainder of the day without having its own budget depleted.

The range of bidding strategies identified above suggests that some advertisers’ budgets are often exhausted while others never. Such heterogeneity in the rate of budget depletion is also reflected in a survey of online advertisers, conducted by the authors. About a third of surveyed advertisers have never had their budgets depleted while another third more than once a week or more. Moreover, industry (Stokes 2010) and academic (e.g., Iyengar et al. 2007, Bu et al. 2010, Zhang and Feng 2011) accounts report on aggressive bidding by some advertisers explicitly aimed at exhausting the budgets of higher-ranked advertisers.

The second result regards the relationship between an advertiser’s budget and its profit. Intuitively, we might expect an advertiser’s profit to be weakly increasing in its budget. While this intuitive outcome arises in our model under certain conditions, we also find that an advertiser’s profit can actually decrease in its own budget. This occurs when both advertisers’ budgets are relatively small. In this case, the second advertiser could use bid jamming to take advantage of the first advertiser’s small budget. But bidding this aggressively with her own small budget opens herself up to being jammed by the first advertiser. In this case, the second advertiser targets the first advertiser’s budget by utilizing the semi aggressive budget-pegging bid strategy so that the second advertiser’s bid is a function of the first advertiser’s budget. Thus, the first advertiser’s CPC, in equilibrium, is positively linked to her own budget while her click volume is not—a situation we refer to as a budget trap. In a budget trap, the first advertiser’s profit is strictly decreasing in her own budget.

Our final result concerns the relationship between the publisher’s revenue and advertisers’ budgets. Again, one might expect the publisher to earn higher revenues when advertisers have larger

---

3 See Appendix A2 for survey details.
budgets. While this is generally the case in our model, there are situations under which publisher revenue decreases in advertisers’ budgets. Recall that when the budget of the first advertiser is small or modest, the second advertiser may employ either a bid jamming or budget pegging strategy as a means to exhaust the first advertiser’s budget. Since this implies a high CPC for the first advertiser, the publisher enjoys high revenue. But, as the first advertiser’s budget increases to an amount which cannot be exhausted, maintaining an aggressive bid strategy by the second advertiser leaves her open to being jammed herself and so she retreats to a conservative strategy, jamming-protection. As a result, the CPC of the first position, and therefore, publisher’s revenue is lower. Our finding of the potential negative revenue impact of budgets provides important implications for publishers who intend to encourage advertisers to expand budgets.

One way that an online publisher might help advertisers increase their budgets is illustrated by the case of Google, who recently provided a new credit card to small- and medium-sized advertisers. Unlike typical credit cards, this credit card can only be used to pay for online advertising expenses on Google. To explore the implications of this practice for publishers, we consider an extension of our basic model with advertisers strategically setting their budgets. Specifically, we allow advertisers to expand their budgets at some cost, which may reflect access to financing or softer liquidity constraints. We find an inverted-U relationship between the publisher’s revenue and the interest rate paid, which measures the opportunity cost of expanding an advertiser’s budget. This result suggests that publishers need to be aware that lower interest rates (and increased budgets) may not always lead to higher bidding (and revenue).

This paper adds to the growing literature of position auctions. Previous studies have examined various topics such as advertisers’ bidding strategies (Edelman et al. 2007, Varian 2007, Börgers et al. 2013), the interplay between organic and sponsored search links (Katona and Sarvary 2010), linking

---

consumers’ optimal search patterns to advertisers’ bidding strategies (Chen and He 2011, Athey and Ellison 2011), the impact of uncertainty on bidding decisions (Athey and Nekipelov 2012), new pricing metrics in online advertising (Zhu and Wilbur 2011, Dellarocas 2012, Liu and Viswanathan 2014), fraudulent behavior (Wilbur and Zhu 2009), the effect of advertisers’ quality on advertising performances (Even-Dar et al. 2007, Jerath et al. 2011, Feng and Xie 2012), the integration of advertising auctions and price competition (Xu et al. 2011), and cyclical bidding patterns (Edelman and Ostrovsky 2007, Zhang and Feng 2011). One common feature of these studies is that they do not account for budget constraints when bidding in position auctions. By introducing budget constraints, our paper finds significant strategic considerations for advertisers’ bidding decisions not identified in earlier work.

Our research is also related to the auction literature with budget-constrained bidders. Che and Gale (1998) study the impact of budget constraints on bidding in standard, single-unit auctions while Benoît and Krishna (2001) study budget-constraints in a sequential second-price auction with two objects. An important difference from these two papers is our focus on the position auction, which has been shown to have different strategic properties than the classic auction mechanisms. More recently, Desai et al. (2014) examined how budgets affect an advertiser’s choice between her own branded keyword and a competitor’s. Sayedi et al. (2014) studied the impact of advertisers’ poaching behavior on their budget allocation between online and offline channels. In contrast to these two papers, we focus on advertiser’s micro-level behavior (bidding decision) under the influence of budget constraints, taking advertisers’ participation in the position auction as given.

Several research papers focus on improving the efficiency of online advertising auctions by following the mechanism design literature. Various results have been shown such as the non-existence of a truthful allocation mechanism with budget-constrained bidders without efficiency loss (Borgs et al. 2014).
2005, Dobzinski et al. 2012), and the existence of an incentive-compatible and efficient mechanism if
budgets are publicly known (Fiat et al. 2011). Our paper differs from this literature by taking the
allocation mechanism as fixed in order to guide advertising managers in online position auctions as
currently practiced.

Our paper is not the first one to recognize the use of aggressive bidding in position auctions.
Ganchev et al. (2007), Lahaie (2006), Zhou and Lukose (2007), Bu et al. (2010), and Iyengar et al.
(2007) all point out that bidders can employ bid jamming in sponsored search and other types of
auctions when bidders are budget constrained. While bid jamming arises in our model as well, our
objective is to fully characterize all types of bid strategies (aggressive, defensive, etc.) in position
auctions with budget-constrained bidders. In this way, our research can provide a more comprehensive
framework to guide advertising managers participating in position auctions.

The rest of the paper proceeds as follows: The next section describes the main model. Section 3
presents the equilibrium bidding strategies and characterizes the conditions under which each type of
strategy arises. We also study the impact of budget sizes on the equilibrium profits of advertisers and
publisher revenue. In Section 4 we extend the model to examine advertisers’ endogenous budget
decisions, while Section 5 provides general discussion and concluding remarks. Proofs of all lemmas,
corollaries, and propositions are relegated to the Appendix.

2. The Model

Our model is a simultaneous-move game of complete information played by two budget-constrained
bidders in a position auction. In this section, we describe the rules and timing of the game and the
bidders’ payoffs. We highlight, as well, the defining assumptions of our model that capture the essential
features of the actual bidding environment we aim to study. Finally, we define the equilibrium concept used to study the game’s outcomes.

We model two focal advertisers competing for two ad positions (aka “slots”) sold by a publisher. We allow advertisers to be different in two dimensions: \((\pi_i, B_i)\), where \(\pi_i\) is the value-per-click and \(B_i\) is the daily budget constraint defined as the daily spending limit in the position auction. Budgets and values-per-click are exogenous and common knowledge to reflect the high level of competitive information available to advertisers.\(^5\)

Denote the bids by the two advertisers as \(b_i\) and \(b_j\). We will assume henceforth that \(\pi_i > \pi_j\) to reflect potential differences in the advertisers’ value-per-click. The advertiser with the highest bid initially wins the first position and pays the second-highest bid, while the second advertiser pays a reservation value \(r\) per-click.\(^6\) The reservation value can be interpreted as the highest bid of a set of non-strategic, lower-ranked advertisers. In this way, the level of \(r\) captures the competitive intensity among other advertisers for this particular keyword. We assume that both advertisers’ values-per-click are sufficiently large that they have incentives to bid in the auction: \(\pi_i > \pi_j > r\).

For convenience, the flow of clicks from consumers is assumed to arrive at a constant rate during the day. Without loss of generality, we normalize the total daily click volume for the second position to be one and denote the click ratio between the first and the second position as \(\sigma\). We assume \(\sigma > 1\) to reflect the positive effect of a higher position on click volume, as documented in previous literature (e.g., Ghose and Yang 2009, Agarwal et al. 2011). Depending on the size of an advertiser’s budget, the volume of clicks can deplete the budget before the end of the day. An advertiser whose budget has been exhausted leaves the auction. If that advertiser, who was initially awarded the first position, has her

---

\(^5\) Later we consider the case in which bidders set their budgets before joining the auction.

\(^6\) Some publishers rank advertisers by the product of advertisers’ bids and their quality scores. While our main model does not consider quality score for simplification, in a separate note which is available upon request, we show that our results are robust to the introduction of quality score: the quality score only shifts the profit functions in a constant rate therefore all the results are quantitatively unchanged.
budget depleted, the second advertiser moves up to the first position, receives the improved click volume, and pays her original CPC, which is \( r \). Figure 1 illustrates the sequence of events when both advertisers’ budgets are exhaustible and advertiser \( i \) at the first position uses up her budget sooner than advertiser \( j \) at the second position. The values \( t_i, t_j \leq 1 \) are defined as the portion of the day the budget of advertiser \( i \) or \( j \), respectively, remains unexhausted. Note that whether advertisers’ budgets are exhausted (i.e., \( t_i, t_j < 1 \)) and the order of their departure (i.e., \( t_i \geq t_j \)) are determined by the advertisers’ budgets and bids. Therefore, we can capture the dynamics above by a one-shot static game of complete information in which advertisers simultaneously submit bids at the beginning of the day.

**Figure 1.** Possible Changes in Position Auction During One Day

Given the exogenous parameters \( \{ \sigma, r, (\pi_i, B_i), (\pi_j, B_j) \} \), bidders obtain the following payoffs as functions of their bids \((b_i, b_j)\):

When \( b_i \geq b_j \), we have advertiser \( i \)'s profit function as

\[
\Pi_i(b_i, b_j) = \begin{cases} 
\sigma(\pi_i - b_j)t_j + \sigma(\pi_i - r)(t_i - t_j) & \text{if } t_i \geq t_j \\
\sigma(\pi_i - b_j)t_i & \text{if } t_i < t_j 
\end{cases}
\]

\[(1)\]

When \( b_i < b_j \),
$$
\Pi_i(b_i, b_j) = \begin{cases} 
(\pi_i - r)t_j + \sigma(\pi_i - r)(t_i - t_j) & \text{if } t_i \geq t_j \\
(\pi_i - r)t_i & \text{if } t_i < t_j
\end{cases}
$$

(2)

where $\Pi_i$ stands for daily advertising return and $t_i/t_j$ is the duration of advertiser $i/j$’s stay in the auction within a day.\(^7\) Suppose advertiser $i$ is initially awarded the first position ($b_i > b_j$). She receives $\sigma t_i$ clicks during her time in the auction. For each click, she receives her value-per-click $\pi_i$ less her CPC, which initially is the second advertiser’s bid, $b_j$. If the second advertiser’s budget is depleted before the day is through ($t_j < 1$) and before the first advertiser’s ($t_j < t_i$), the first advertiser’s CPC reduces to $r$ for the remainder of her time in the first position ($t_i - t_j$). Otherwise, her CPC is $b_j$ for the entire duration, $t_i$, in the auction.

Alternatively, suppose advertiser $i$ is initially in the second position ($b_i < b_j$). For each click she receives $\pi_i$ less her CPC, which is $r$ for the entire time in the auction. If the first advertiser’s budget is not exhausted before the end of the day ($t_j = \min\{\frac{b_j}{\sigma b_i}, 1\} = 1 \geq t_i$), advertiser $i$ receives a total of $t_i$ clicks. Otherwise, if $t_j < t_i$, the advertiser $i$ occupies the second position and receives $t_j$ clicks and ascends to the first position at time $t_j$, where she receives $\sigma(t_i - t_j)$ clicks.

One analytical challenge of studying position auctions with budget-constrained bidders is the existence of multiple equilibria. While in standard second-price auctions, the unintuitive equilibria can usually be refined by finding weakly dominant strategies. However, in our setting this is not sufficient.\(^8\) Therefore, we employ the notion of an Undominated Nash Equilibrium (UNE). The UNE is an equilibrium concept established in Palfrey and Srivastava (1991) and has been extensively used in previous auction literature (e.g., Benoît and Krishna 2001, Börgers et al. 2013) to select the most intuitive equilibrium.

\(^7\) For the detailed characterization of advertisers’ profits, please see Table B2 in Appendix B.

\(^8\) Additional details are available from the authors upon request.
**Definition 1.** The pair of bids \((\tilde{b}_i, \tilde{b}_j)\) is a UNE if

1. Nash: \(\Pi_i(\tilde{b}_i, \tilde{b}_j) \geq \Pi_i(b_i, \tilde{b}_j)\) for all \(b_i\) and \(\Pi_j(\tilde{b}_i, \tilde{b}_j) \geq \Pi_j(b_i, b_j)\) for all \(b_j\).

2. Undominated: There does not exist any \(\tilde{b}_i\) such that \(\Pi_i(\tilde{b}_i, b_j) \leq \Pi_i(b_i, b_j)\) for all \(b_j\) nor does there exist any \(\tilde{b}_j\) such that \(\Pi_i(b_i, \tilde{b}_j) \leq \Pi_i(b_i, b_j)\) for all \(b_i\).

This definition requires that a pair of strategies has (1) the usual Nash property and (2) a mild refinement that requires players to never use a weakly dominated strategy in equilibrium. In our context of position auctions, a pair of bids is a UNE if it is, first a Nash equilibrium and second, each advertiser’s bid is not weakly dominated by any other possible bid. In this way, this refinement screens out all implausible equilibria. One additional advantage of employing the UNE in our analysis is its unique prediction of the equilibrium outcome, which includes the CPC, positions, and durations in the auction for both advertisers.

**3. Equilibrium Analysis**

In this section, we derive our main results by studying the equilibrium properties of the game described above. Let \((\tilde{b}_i, \tilde{b}_j)\) be the UNE corresponding to a budget for any \((B_i, B_j)\) \(\in \Theta = \{ (B_i, B_j) \in \mathbb{R}^2 | B_i, B_j > 0 \}\). We provide a full characterization of bidding strategies throughout the entire budget-space \(\Theta\). Since the full details of this characterization are quite involved, we provide here a categorization of advertisers’ equilibrium bidding strategies based on their strategic motivations: Aggressive (bid jamming), Defensive (jamming protection), and Semi-aggressive (budget pegging). These motivations depend on the level of outside competition for the auction’s keyword, as measured by the reservation value \(r\). As we show in section 3.1, if outside competition is relatively low \((r < \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\})\), advertisers engage only in aggressive and defensive strategies. However, as the level of...
outside competition increases, an advertiser may also be motivated to employ a semi-aggressive bidding strategy, which we show in section 3.2. Then, we use these results to study the impact of budget sizes on advertisers’ profits and on the publisher’s revenue in section 3.3.

3.1. Bidding Equilibrium with a Low Reservation Value

When the reservation value is below the threshold, \( r < \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\} \), advertisers are free from the threat of budget exhaustion by the reservation value \( r \) because both of them can afford the highest possible daily expenses \( \sigma r \). This implies that only the first advertiser is possibly to be forced out from the auction during a day.

Because bidding without budgets is well-studied, this case serves a useful benchmark for the subsequent analysis.

Lemma 1. If both advertisers are not budget-constrained \((B_i = B_j = \infty)\), then the UNE is \( \bar{b}_i^* = (1 - \frac{1}{\sigma}) \pi_i + \frac{r}{\sigma} \) and \( \bar{b}_j^* = (1 - \frac{1}{\sigma}) \pi_j + \frac{r}{\sigma} \).

Since advertisers’ profits are independent of budgets in this case, the equilibrium reflects the weakly dominant bids in a standard second-price auction: both advertisers choose a bid under which her profit is equalized across the two positions when the rival ties to this bid. We denote \( b_i^* \), \( b_j^* \) as budget-free bids and the budget-free region as the set \( \Theta_{BF} \) as the set of budgets \((B_i, B_j) \in \Theta \) in which \( \bar{b}_j = b_j^* \) is part of the UNE. Generally, budgets in the budget-free region are sufficiently large that they do not play an important strategic role in bidding decisions.

Proposition 1a. Let \( r < \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\} \). For \((B_i, B_j) \in \Theta_{BF}\) where both advertisers’ budgets are sufficiently large, the UNE is \( \bar{b}_i > \bar{b}_j = b_j^* \) and both advertisers keep their positions for the entire auction. (Exact expressions for equilibrium bids and \( \Theta_{BF} \) are in the appendix.)
The UNE predicts that the low-value advertiser exactly submits the budget-free bid and there exists a bounded set of undominated bids for the high-value advertiser at the first position. Intuitively, the first advertiser will bid below an upper bound to ensure the inexhaustibility of her own budget and bid above a lower bound under which her profit is indifferent between two positions. Note that the equilibrium outcome is unaffected by the first advertiser’s bid when both advertisers’ budgets are large enough to be inexhaustible: \((B_i, B_j) \in \Theta_{BF}\). Hence, \(\Theta_{BF}\) corresponds to the benchmark case because both advertisers’ CPC and click volume are the same as if they faced no budget constraints.

We now consider the regions of budgets outside of \(\Theta_{BF}\) in which at least one of the advertiser’s budget is small enough that it could be exhausted during the auction’s duration. As we now show, the equilibrium bids will always depend on at least one advertiser’s budget. We partition the region of \(\Theta \setminus \Theta_{BF}\) into two sets. One set, \(\Theta_A\) is the region in which neither advertiser has a significant budget advantage over the other. In this case, the UNE always reflects aggressive bidding by the advertiser in the second position, and we refer to this region as \(\Theta_A\), where “A” stands for “aggressive”. The other set, \(\Theta_D\) is the region in which one advertiser has a distinct budget advantage over the advertiser. In this case, the UNE always reflects defensive bidding by the advertiser with the smaller budget and we refer to this region as \(\Theta_D\), where “D” stands for “defensive”. To further characterize the equilibrium outcomes in each region, we subscript \(\Theta_A\) and \(\Theta_D\) by the advertiser who is initially assigned the first position. Figure 2 provides an overview of the entire collection of budget regions.

We start with the UNE for budgets in \(\Theta_A\).

**Proposition 1b.** Let \(r < \min \left\{ \frac{B_i}{\sigma_1}, \frac{B_j}{\sigma_1} \right\}\). For \((B_i, B_j) \in \Theta_A = \Theta_{Al} \cup \Theta_{Aj}\), the UNE is characterized by bid jamming, with the second-ranked advertiser bidding just below (small \(\varepsilon > 0\)) the first-ranked advertiser. (Exact expressions for equilibrium bids and \(\Theta_A\) are in the Appendix.)
(i) \[ \Theta_{Ai}: \text{If } \frac{\pi_i B_i + \left(1 - \frac{1}{\sigma}\right)(\pi_i - r)B_j}{B_i + \sigma(\pi_i - r)} > \frac{\pi_j B_j + \left(1 - \frac{1}{\sigma}\right)(\pi_j - r)B_i}{B_j + \sigma(\pi_j - r)} \] or \( B_i \) is moderate, then \( \bar{b}_i < b_i^* \) and \( \bar{b}_j = \bar{b}_i - \varepsilon \). Bidder \( i \) is in the first position until her budget is depleted at \( t = \frac{B_i}{\sigma b_i} < 1 \).

(ii) \[ \Theta_{Aj}: \text{Otherwise, } \bar{b}_j < b_j^* \text{ and } \bar{b}_i = \bar{b}_j - \varepsilon. \text{ Bidder } j \text{ is in the first position until her budget is depleted at } t = \frac{B_j}{\sigma b_i} < 1. \]

Figure 2. Bidding Equilibrium with a Low Reservation Value

The second-ranked advertiser is induced to bid more aggressively when the difference in advertisers’ budgets is small (the lower portion of \( \Theta_{Ai} \) and all of \( \Theta_{Aj} \). This is because the small discrepancy in budgets intensifies the competition for the first position, which accentuates bidders’ motivation for bid jamming strategies. Under this competitive scenario, either the high-value or the low-value advertiser might use bid jamming, depending on their relative budget sizes determined by the
comparison between \( \frac{\pi_i B_i + (1-\frac{1}{2})(\pi_i-r)B_j}{B_i + \sigma(\pi_i-r)} \) and its counterpart. Here \( \frac{\pi_i B_i + (1-\frac{1}{2})(\pi_i-r)B_j}{B_i + \sigma(\pi_i-r)} \) represents advertiser i’s willingness to pay for the first position when both advertisers’ budgets are potentially exhaustible.

We also find that bid jamming occurs when the low-value advertiser \( j \) has a larger budget and the high-value advertiser i’s budget is moderate (the upper portion of \( \Theta_{D_1} \)). As advertiser i’s bid is positively associated with her budget, this makes it too costly for advertiser \( j \) to outbid advertiser i to stay at the first position in this situation.\(^9\) Instead, advertiser \( j \) prefers to bid jam at the second position. Meanwhile, the budget advantage of advertiser \( j \) thwarts advertiser i’s incentive to bid jam, resulting in advertiser i’s staying at the top position. We now turn to \( \Theta_D \).

**Proposition 1c.** Let \( r < \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\} \). For \( (B_i, B_j) \in \Theta_D = \Theta_{D_1} \cup \Theta_{D_2} \), the UNE is characterized by jamming protection, with the second-ranked advertiser bidding at a conservative level under which her profit is unaffected by the use of bid jamming by the rival. (Exact expressions for equilibrium bids and \( \Theta_D \) are in the Appendix.)

(i) \( \Theta_{D_1} \): If \( B_i > B_j \) then \( \bar{b}_i > \bar{b}_j \) and \( \bar{b}_j = \frac{\pi_i B_j}{\pi_i + B_j - r} < b_j^* \) so that bidder i is in the first position, bidder \( j \) in the second, and budgets are never depleted.

(ii) \( \Theta_{D_2} \): If \( B_i < B_j \) then \( \bar{b}_i = \frac{\pi_i B_i}{\pi_i + B_i - r} < b_i^* \) and \( \bar{b}_j > \bar{b}_i \) so that bidder i is in the second position, bidder \( j \) in the first, and budgets are never depleted.

Jamming protection is used by the second advertiser when one advertiser’s budget is small and the other’s budget is much larger. As the first-ranked advertiser’s budget advantage exceeds a certain threshold, the second advertiser cannot benefit from bid jamming. The inexhaustibility of the first advertiser’s budget and the vulnerability of the second advertiser’s budget cause the second advertiser to shade her bid below \( b^* \). This is reflected by the second advertiser’s decrease in bid when the first

\(^9\) The formal proof for the positive relationship between advertisers’ bids and budgets can be found in Corollary D1 in Appendix D.
advertiser’s budget surpasses the boundary of \( \Theta_A \) and \( \Theta_D \), or more specifically, when an increase in \( B_i \) shifts the regime from \( \Theta_{Ai} \) to \( \Theta_{Di} \) or an increase in \( B_j \) shift the regime from \( \Theta_{Aj} \) to \( \Theta_{Dj} \) in Figure 2. In equilibrium, the second advertiser employs a defensive strategy by bidding at \( \bar{b} = \frac{\pi B}{\pi + B - r} \), which shields her profit from being hurt by the potential bid jamming used by the rival. To see this, in part (i) advertiser \( j \)'s profit at the second position is \( \pi j - r \), which equals her profit \( \frac{\pi j B_j}{B_j} - B_j \) when the rival advertiser \( i \) employs bid jamming. Notice that the second advertiser’s bid is always below her budget-free bid in \( \Theta_D \) because the potential exhaustibility of her budget drives her to bid more cautiously.

Finally, the second advertiser’s conservative bid provides more incentive for the large-budget advertiser to stay at the first position. This explains why the large-budget advertiser always outbids the small-budget advertiser in this case.

### 3.2. Bidding Equilibrium with a High Reservation Value

We now investigate the situation in which the reservation value is large so that \( r \geq \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\} \). In this case, the external competition in the position auction is sufficiently intense that at least one advertiser will surely exit the auction before the end of the day. Furthermore, due to the large reservation value, the advertiser at the second position might not have enough budget to digest the additional click volume should she ascend to the first position before the end of the day. This consideration gives rise to a new bidding strategy, not present in the case considered earlier, which we call *budget pegging*. Budget pegging is a semi-aggressive bid strategy used by the second-ranked advertiser who aims to slowly exhaust the budget of the above-ranked advertiser. We denote the budget region \( \Theta_p \) as the set of budgets \( (B_i, B_j) \) such that *budget-pegging* is part of a UNE, with \( \Theta_{pi} \) and \( \Theta_{pj} \) indicating whether advertiser \( i \) or \( j \) is initially in the first position. Aggressive and defensive strategies arise in the UNE in this case as well.
and we maintain the set notation $\Theta_A$ and $\Theta_D$ used earlier. Finally, there is a region of budgets, denoted by $\Theta_R$, in which at least one advertiser’s budget is so small that it is inevitably exhausted before the end of the auction period regardless of her position. In this case, the second advertiser always bids $r$. Since there is no interesting interactions among advertisers for $(B_i, B_j) \in \Theta_R$, we henceforth ignore the discussion of this region. See Figure 3 for a graphical depiction of these equilibria regimes in the case of large $r$ and Proposition 2 for a characterization.

**Proposition 2.** Let $r \geq \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\}$. In addition to bid jamming in $\Theta_A = \Theta_{Ai} \cup \Theta_{Aj}$ and jamming protection in $\Theta_D = \Theta_{Di} \cup \Theta_{Dj}$, the UNE in $\Theta_F = \Theta_{Pi} \cup \Theta_{Pj}$ is characterized by budget pegging, with the second-ranked advertiser bidding in a fixed proportion to the first-ranked advertiser’s budget. (Exact expressions for equilibrium bids and budget regions are in the appendix.)

\[
(i) \quad \Theta_{Pi}: \text{If } \frac{\pi_i B_i + \left(1 - \frac{1}{\sigma}\right)(\pi_i r) B_j}{B_i + \sigma(\pi_i r)} > \frac{\pi_j B_j + \left(1 - \frac{1}{\sigma}\right)(\pi_j r) B_i}{B_j + \sigma(\pi_j r)} \text{ and } B_j \text{ is relatively small, then } \bar{b}_i > \bar{b}_j \text{ and } \\
\bar{b}_j = \frac{(\sigma - 1) r B_i}{\sigma (\sigma r - B_j)} \text{ Bidder } i \text{ is in the first position until her budget is depleted at } t_i = \frac{\sigma (\sigma r - B_j)}{(\sigma - 1) r} < 1 \text{ and bidder } j \text{’s budget is depleted at } t_j = 1.
\]

\[
(ii) \quad \Theta_{Pj}: \text{If } \frac{\pi_i B_i + \left(1 - \frac{1}{\sigma}\right)(\pi_i r) B_j}{B_i + \sigma(\pi_i r)} < \frac{\pi_j B_j + \left(1 - \frac{1}{\sigma}\right)(\pi_j r) B_i}{B_j + \sigma(\pi_j r)} \text{ and } B_i \text{ is relatively small, then } \bar{b}_i = \frac{(\sigma - 1) r B_j}{\sigma (\sigma r - B_i)} \text{ and } \bar{b}_j > \bar{b}_i \text{ Bidder } j \text{ is in the first position until her budget is depleted at } t_j = \frac{\sigma (\sigma r - B_i)}{(\sigma - 1) r} < 1 \text{ and bidder } i \text{’s budget is depleted at } t_i = 1.
\]
The budget-pegging strategy is employed by the second advertiser under two scenarios: i) the first-ranked advertiser has a relatively small budget that is exhaustible before the end of the period; ii) the second-ranked advertiser’s budget is also small so that she can only afford a fraction of the total click volume in the first position by paying the reservation value $r$ per click upon the ascension. In this situation, the second advertiser does not benefit from the aggressive jamming strategy because driving out the first advertiser sooner brings no extra benefit and exposes her to being jammed by the rival. Instead, the second advertiser bids semi-aggressively so that her budget is exactly used up at the end of the period. Because the second advertiser’s bid in this situation is linked to the first advertiser’s budget, the budget-pegging region $\Theta_p$ has special significance and gives rise to a scenario we discuss in the next section called a budget trap. In a budget trap, the first advertiser’s revenue actually decreases in her own budget. We now provide a simple example below to help understand the second advertiser’s bidding behavior in this case.
Example. Assume \( r = 1, \sigma = 3 \) and advertiser \( j \)'s budget is 2, which implies that the maximum click volume for advertiser \( j \) is 2 determined by \( \frac{B_j}{r} \). Advertiser \( j \)'s best strategy is to choose a bid so that she spends 50% time at the second and the first position respectively to receive the maximum click volume. This is optimal because lowering the bid to stay longer at the second place leaves unused budget for advertiser \( j \) and thereby reduces the click volume, while bidding higher to ascend to the first position earlier does not result in more clicks but increases the threat of being jammed by the competitor.

We argued in the previous example that the second advertiser should select a minimum bid that her budget is depleted right at the end of the auction. And since her budget is large enough to not be depleted in the second position only, the second advertiser must ascend to the first position and do so with precise timing. As our example shows, advertiser \( j \)'s maximum click volume, and therefore the optimal timing decision, is only dependent on her own budget and the reservation value, which leads to the pegging strategy: the second advertiser pegs on the first advertiser's budget to make sure she reaches the first position at the right time.

Our equilibrium analysis indicates one additional finding that offers a new perspective on a phenomenon known as the “position paradox” (Jerath et al. 2011). In the position paradox, the low-value advertiser occupies the top position while the high-value advertiser is allocated the lower position. This outcome occurs in the budget regions where advertiser \( j \) is in the first position (e.g., \( \Theta_{Aj} \cup \Theta_{Dj} \) in Figure 2 and \( \Theta_{Aj} \cup \Theta_{Dj} \cup \Theta_{Pj} \) in Figure 3). Jerath et al (2011) attest this flip in positions to the difference between advertisers’ click-through rates. In our model, however, advertisers have the same click-through rates for any given position. The emergence of the position paradox in our framework, therefore, is completely driven by advertisers’ concerns about budget constraints. Because of a small budget, the high-value advertiser may find it more profitable to stay at the lower position than at the higher position with limited duration. As for the low-value advertiser with a larger budget, she tends to bid more
aggressively for the first position because her budget advantage provides her a safety net to prevent 
budget depletion. This situation plays an important role in revenue analysis of the next section.

3.3. Revenue Analysis

We now examine the revenue impact of daily budgets in this section. For detailed illustration of the 
publisher’s revenue and advertisers’ profits in each budget area, please see Table E1 in Appendix E.

**Proposition 3.** If \( r < \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\} \) then each advertiser’s profit is weakly increasing with her budget. If \( r \geq \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\} \) then each advertiser’s profit is also weakly increasing with her budget except in 
region \( \Theta_p \), in which the first advertiser’s profit is strictly decreasing with her own budget.

When advertisers’ budgets are inexhaustible by the reservation value (i.e., \( B_i, B_j \geq \sigma r \)), our 
results suggest that a larger daily budget can bring advertisers an additional advantage in position 
auctions. This is because as the external competitive pressure becomes attenuated, advertisers’ profits 
are mainly affected by the intensity of internal competition as measured by the discrepancy in budgets. 
A larger budget provides a safety net for advertisers to avoid budget depletion and thus encourages them 
to bid more aggressively to exhaust the budget of an above-ranked competitor. Furthermore, when an 
advertiser’s budget exceeds a threshold beyond which the below-ranked competitor switches from the 
strategy of bid jamming to jamming protection, the first advertiser’s profit is strictly improved because 
of the lower CPC. This explains why advertisers benefit from a larger budget constraint in the case with 
low external competition (\( r < \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\} \)).

When the intensity of external competition is high (\( r \geq \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\} \)), bid tactics are no longer 
determined solely by the internal rivalry between advertisers \( i \) and \( j \). As we showed in Proposition 2, the 
second-ranked advertiser engages in budget pegging in \( \Theta_p \) where the strength of both internal and
external competition is high. Next we explain why the first advertiser’s profit strictly decreases with her budget in region $\Theta_p$. Consider $(B_i, B_j) \in \Theta_p$ where advertiser $j$ pegs on advertiser $i$’s budget. To implement the budget pegging, advertiser $j$ must make sure to exhaust advertiser $i$’s budget at a fixed time point by bidding proportionately to $B_i$. This links advertiser $i$’s budget directly to her own CPC in equilibrium. In other words, her CPC increases in her own budget but her total click volume, $\frac{\sigma(\sigma r-B_j)}{(\sigma-1)r}$, does not. We call this situation a budget trap because an increase in the first advertiser’s budget results only in a higher cost and therefore a lower profit.

Next we summarize our finding of the publisher’s revenue.

**Proposition 4.** The publisher’s revenue $\Pi_p$ weakly increases with advertisers’ budgets except in the following two scenarios.

(i) Change in bid strategy: an increase in the first-ranked advertiser’s budget causes the second advertiser to switch from bid jamming to either jamming protection or budget-free bid.

(ii) Change in ad position: an increase in the high-value advertiser’s budget, $B_i$, causes her to obtain the first position instead of the second position in equilibrium.

It is intuitive to think that an increase in advertisers’ budgets should always benefit the publisher. Since a large budget increases the time an advertiser stays in the auction, more surplus is transferred to the publisher. However, after accounting for advertisers’ strategic interactions, a counterintuitive result arises: the publisher’s revenue can strictly decrease with advertisers’ budgets. This can happen in two scenarios. The first scenario is when the first advertiser has a more-than-necessary daily budget to deter the below-ranked advertiser’s incentive to bid aggressively. In this case, the first advertiser’s budget is no longer exhaustible even if the second advertiser engages in bid jamming. As a response, the second advertiser reduces her bid and becomes less aggressive. This reduces the first advertiser’s payment and
therefore hurts the publisher’s revenue. The situation can be seen graphically in Figures 2 & 3 in which an increase in budgets causes a switch out of $\Theta_A$ and into $\Theta_D$.

The second scenario is when an increase in the high-value advertiser’s budget induces her to bid for the first position despite a disadvantage in budgets. This corresponds to the case when an increase in $B_i$ causes a switch from $\Theta_{A_j}$ to $\Theta_{A_l}$ (for any $r$), from $\Theta_{D_j}$ to $\Theta_{A_l}$ (only for $r < \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\}$) or from $\Theta_{P_j}$ to $\Theta_{P_l}$ (only for $r \geq \min \left\{ \frac{B_i}{\sigma}, \frac{B_j}{\sigma} \right\}$). In Figures 2 & 3, these are crossings across the boundary indicated by the dotted line. Note that in all of these situations, the publisher benefits from the larger-budget advertiser staying in the first position because of the greater surplus transferred from advertisers’ budgets to the publisher. Hence, when the high-value advertiser’s budget increases to the level that forces the low-value but larger-budget competitor to step down from the top, the publisher suffers revenue losses.

Proposition 4 also brings up the issue that the publisher can play an active role in position auctions. Because budget constraints can affect the publisher’s revenue, the publisher may have incentive to limit the advertisers’ information set in order to reduce the strategic bidding.\footnote{We thank a reviewer for pointing this out.} Incorporating the publisher’s strategic incentive into our model is beyond the objective of current paper, but points to an interesting direction for future research.

4 Endogenous Advertising Budgets

As the previous section pointed out, enhanced budgets do not always generate higher revenue for the publisher. Yet, some publishers intend to encourage advertisers to expand budgets by providing financial services with a competitive interest rate (e.g., Google’s credit card). To understand the implication of this practice, we relax the assumption of fixed budget and allow the advertisers to set
budgets before joining the auction. Before discussing the endogenous advertising budgets, it is helpful to first introduce a concept of a sufficient budget.

**Definition 2.** An advertiser $i$’s daily budget is *sufficient* if it is inexhaustible under her highest willingness to pay for the first position, that is, $B_i \geq B_i^* \equiv \sigma b_i^*$.

By Proposition 3, if advertisers are not in region $\Theta_p$ and can costlessly increase budgets, they should always set daily budgets above the sufficient threshold. However, an expansion of an advertising budget typically requires the firm to divert spending from other investments (Che and Gale 1998). The opportunity cost for the advertising budget expansion is defined as the highest net return from outside options such as investments in financial markets or other advertising channels.

We consider a two-stage game to model budget decisions made by advertisers. In the first stage, advertisers simultaneously choose their advertising budgets before participating in the position auction. In the second stage, both advertisers’ budgets become common knowledge and advertisers compete on bids as shown in our previous model. The rationale for this two-stage setting is due to the much higher frequency of the change in bids made by advertisers than the frequency of change in budgets. We assume that the marginal cost of budget expansion is symmetric and constant, that is, $C(B) = cB$ where $c > 0$ refers to the marginal cost of the budget expansion.

We assume that advertisers now face a softer budget constraint: they might either choose a sufficient budget $B_i^H = B_i^*$ to obtain the greatest budget power in the position auction or a smaller budget $B_i^L = \alpha B_i^*$, for some positive $\alpha < 1$. We assume the reservation value $r$ to be zero for tractability. Under this setting, advertisers need to balance the trade-off between the higher cost of setting a sufficient budget in the first stage and the improved advertising profit in the second stage. We solve the game backwards and report our findings in the following proposition.
**Proposition 5.** When $\alpha$ is relatively small, advertiser $i$’s budget weakly decreases in $c$, whereas advertiser $j$’s budget and the publisher’s revenue are in an inverted-U relationship with $c$. In other words, a decrease in $c$ can lead to a decrease in the publisher’s revenue despite an increase in advertiser $i$’s choice of budget.

We find that the budget choices of two advertisers are different with respect to the marginal budget cost (Figure 4). As expected, high-value advertiser $i$’s budget weakly decreases with the cost parameter $c$. However, low-value advertiser $j$’s budget can increase with $c$ when $c$ is below the threshold $c_2$, under which advertiser $j$ is indifferent between two budget options conditional on $i$ setting the low budget ($c_1$ is the similar threshold for advertiser $i$). The underlying reason is that advertiser $j$ has more incentive to set a high budget than advertiser $i$. Unlike advertiser $i$ who has the advantage to win the first position conditional on both advertisers’ choosing the same budget, advertiser $j$ cannot enjoy the greater click volume from the first position unless she sets a high budget. Thus, advertiser $j$ is more willing to increase her budget and gain an advantage in the bidding stage. This argument is supported by $c_1 < c_2$ in Figure 4.\(^{11}\)

\[\text{Figure 4: Budget Choices of Advertisers} \]

![Diagram](image)

(a) when $\frac{\pi_i}{\pi_j}$ is small

(b) when $\frac{\pi_i}{\pi_j}$ is large

\(^{11}\)The exact expressions for $c_1$ and $c_2$ can be found in Appendix F.
Figure 4. Relationship between Budget Choices and Marginal Opportunity Cost

Notes. The dash line stands for the high-value advertiser’s budget and the solid line stands for the low-value advertiser’s budget. In case (b), only a mixed-strategy equilibrium exists when \( c \in [c_1, c_3] \), therefore we denote \( E(B) \) as the expected budget of advertisers in this case.

Now we can explain how the equilibrium budgets of advertisers vary with the marginal cost \( c \).

When \( c \) is very small, advertiser \( i \) has a dominant strategy to choose the high budget because the cost is far below the gain from the advertising profit. Consequently, advertiser \( j \)’s best response is to select the low budget because she cannot benefit from a high budget when the competitor’s budget is sufficient. As \( c \) increases, advertiser \( i \) is more likely to switch to the low-budget option because of the increased cost associated with the high budget. Meanwhile, advertiser \( j \) is inclined to take advantage of advertiser \( i \)’s increased likelihood of setting a low-budget constraint. When \( c \) rises further, both advertisers strictly prefer the low-budget option because the marginal cost outweighs the marginal benefit of the high-budget option.

The counterintuitive finding that the publisher’s revenue can increase with the marginal budget cost can be explained by the following. As Proposition 4 indicates, the publisher’s revenue might increase when advertiser \( i \)’s budget drops below the sufficient threshold because advertiser \( j \) is more likely to use a budget-depletion tactic (bid jamming or budget-pegging), which transfers a larger surplus from advertiser \( i \)’s budget to the publisher. Here advertiser \( j \)’s incentive to bid aggressively is further strengthened because not only is advertiser \( i \)’s budget decreasing, but also because the budget difference between them is shrinking. This explains the increasing part of the inverted-U relationship. As the marginal budget cost exceeds a certain threshold, the publisher’s revenue falls because both advertisers invest only a small amount in the position auction. In the end, we can see when \( c \) rises from \( [c_1, c_2] \), advertiser \( i \) lowers the budget while the publisher’s revenue increases.
5. Discussion and Implications

In this section, we discuss how results yielded from our analysis above lead to several managerial insights for both advertisers and publishers of online advertising.

5.1. A Strategic Role of Budgets for Advertisers

Our analysis suggests that advertisers bidding in position auctions with budget constraints face complex strategic considerations. Advertisers should recognize that equilibrium bidding can be dramatically different with budget constraints than without. Furthermore, bidding strategies heavily depend on the relative size of both advertisers’ budgets. For example, staying at the first position without a large budget can be risky since the lower advertiser may bid aggressively in order to overtake the first position later in the auction.

Advertisers should also be aware of a peculiar “budget trap” situation, in which an advertiser’s profit may actually decrease with an increase in her own budget. Our analysis indicates that a budget trap will arise when both advertisers’ budgets are relatively small and the external competition is fierce. Advertisers may detect this situation by checking two conditions: 1) the daily budget is nearly always depleted and 2) an increase in the daily budget does not bring significantly more clicks. If both conditions are met, advertisers are subject to the adverse consequence of budget expansion.

5.2. Caveats on Budget-related Policies for Publishers

One major finding in this paper is that the publisher’s revenue can be negatively affected by advertisers’ budgets in position auctions. Thus, publishers should be careful when encouraging advertisers to expand budgets by offering any sort of financial services. For example, Google recently offered small-sized advertisers a new credit card with a competitive interest rate and ample credit line to promote larger budgets in position auctions. This credit card can only be used to pay for online advertising expenses and is exclusively offered to small advertisers selected by Google. This practice can be regarded as an
effort to encourage advertisers to allocate more money to online advertising. However, as indicated in Proposition 4, a small-sized firm with a high value-per-click but a small budget may induce aggressive bid jamming by a lower ranked advertiser. Easing the budget of the high value-per-click advertiser may discourage the aggressive bidding and lower the CPC and, correspondingly, publisher revenue. Therefore, publishers may want to offer budget assistance selectively, to low value-per-click advertisers. Furthermore, our finding of the inverted-U relationship between the publisher’s revenue and the marginal budget cost implies that publishers should be aware that there exists a set of optimal interest rates to charge advertisers if they plan to extend such financial services to all advertisers.

5.3. Limitations and Directions for Future Research

In our analysis, we made several modeling assumptions and focused on the basic incentives of budget constrained bidders in a position auction. To keep the analysis tractable, we limited ourselves to two advertisers bidding for two positions because the number of equilibrium outcomes increases exponentially with the number of positions due to the possibility of strategic bidding. This technical constraint prevented us from studying the impact of additional advertisers bidding for more than two slots. Extending our model in this way may lead to new insights regarding advertisers’ decisions to enter the auction and the impact of additional bidders on profits and publisher revenues. However, we do not expect the increase in the number of competing advertisers to qualitatively change the fundamental strategic motivations of bidders we identified here. This is because even in a highly competitive case, as measured by a large reservation value $r$, we find that the publisher’s revenue can still decrease with advertisers’ budgets.

Although we focused on the case where advertisers know rivals’ budgets and values-per-click, there may be situations in which advertisers do not. In Appendix G1, we studied an extension of our

---

12 In a position auction with $N$ ad positions and advertisers, the equilibrium outcome could be in $2^{N-1}/N!$ different forms due to the $N!$ possible types of ranks and potential occurrence of strategic bidding between each pair of adjacent advertisers.
model in which advertisers have uncertainty regarding their budget constraints and showed that our main result regarding the negative impact of budgets on the publisher’s revenue still holds. Future research can further extend to the case where advertisers are also uncertain about competitors’ value-per-click. This extension, however, requires modeling a two-dimensional, incomplete information game for position auctions - an interesting but challenging opportunity for future work.

In our main model the position outcome is completely determined by the order of advertisers’ bids. However, some publishers may rank ads not only based on an advertiser’s bid but also on stochastic elements. We investigate this situation in Appendix G2 and confirm that advertisers’ budget can decrease publisher’s revenue. But obviously more work is needed to better understand the implications of the practice of randomizing rank in advertising auctions.

We also assumed that the budgets submitted by advertisers are real financial constraints. This simplification allows us to abstract away any strategic gaming by budget misreporting in order to focus on advertisers’ bid strategies. Future research can extend our study by considering whether/ how this strategic decision may affect advertisers’ incentive to bid.

Finally, as mentioned in section 3.3, we focused on understanding the role of budget constraints in advertisers’ bidding strategies in a stylized GSP setting. Therefore we did not consider the strategic role of publisher in the model. A full analysis of the publisher’s incentive in GSP auction is beyond the objective of this paper, but it may point to a profitable direction for future research.
Appendix A.

A1. Empirical Evidence for the Influence of Budget on Advertisers’ Positions

This section provides empirical support to the claim in the introduction that advertisers’ budgets are a factor in determining their ranking in position auctions. We analyzed data from Spyfu, a privately held advertising-services firm that specializes in search-advertising data. This dataset includes average daily budgets, average ad positions and the number of paid keywords for 11 firms that advertise on Google from Mar 2012 to Jan 2013 (Table A1 lists these firms). The 11 firms we considered belong to five product categories, which are selected based on two criteria: 1) categories with a short product line so that firms tend to use a narrower set of keywords and; 2) categories with firms that spend significantly on Google search advertising over time.

Our dataset is on a monthly basis and aggregated to the campaign level for each firm. Thus, we compute the average daily budget per keyword for each firm by dividing the average daily budget by the number of paid keywords. Using the monthly data for 11 months, we calculate the correlation between the average daily budget per keyword and average ad position for each firm. We report the summary statistics and correlation results in Table A1.

Table A1. Summary Statistics and Correlation Results

<table>
<thead>
<tr>
<th></th>
<th>Avg Daily Budget (1k$)</th>
<th>No. of Keywords (1k)</th>
<th>Avg DB Per Keyword ($) V1</th>
<th>Avg Ad Position V2</th>
<th>Corr. btw V1 &amp; V2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Game console</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PlayStation</td>
<td>0.46 (0.21)</td>
<td>1.55 (1.57)</td>
<td>1.96 (2.62)</td>
<td>3.64 (0.96)</td>
<td>-0.54</td>
</tr>
<tr>
<td>Xbox</td>
<td>0.64 (0.47)</td>
<td>0.85 (0.43)</td>
<td>1.01 (0.72)</td>
<td>3.72 (1.55)</td>
<td>-0.47</td>
</tr>
<tr>
<td>Nintendo</td>
<td>1.41 (1.17)</td>
<td>2.46 (3.02)</td>
<td>0.93 (0.62)</td>
<td>3.18 (1.16)</td>
<td>-0.58</td>
</tr>
<tr>
<td><strong>Helmet camera</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GoPro</td>
<td>6.72 (3.63)</td>
<td>1.90 (0.55)</td>
<td>4.31 (3.46)</td>
<td>2.48 (0.77)</td>
<td>-0.56</td>
</tr>
<tr>
<td>Contour</td>
<td>0.36 (0.20)</td>
<td>0.21 (0.03)</td>
<td>1.69 (0.90)</td>
<td>3.45 (0.54)</td>
<td>-0.63</td>
</tr>
<tr>
<td><strong>Auto GPS</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Garmin</td>
<td>2.73 (2.76)</td>
<td>3.62 (0.91)</td>
<td>0.68 (0.54)</td>
<td>4.65 (0.27)</td>
<td>-0.14</td>
</tr>
<tr>
<td>Tomtom</td>
<td>0.42 (0.19)</td>
<td>0.39 (0.25)</td>
<td>1.41 (0.93)</td>
<td>3.44 (1.39)</td>
<td>-0.54</td>
</tr>
<tr>
<td><strong>Tax-prep software</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TurboTax</td>
<td>14.84 (17.55)</td>
<td>10.17 (7.32)</td>
<td>1.21 (0.81)</td>
<td>3.90 (0.79)</td>
<td>0.06</td>
</tr>
<tr>
<td>H&amp;RBlock</td>
<td>12.45 (15.64)</td>
<td>6.63 (4.67)</td>
<td>2.67 (3.31)</td>
<td>3.71 (1.15)</td>
<td>-0.68</td>
</tr>
<tr>
<td><strong>Single-cup coffee brewer</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Keurig</td>
<td>2.31 (1.51)</td>
<td>2.03 (0.62)</td>
<td>1.24 (0.96)</td>
<td>5.44 (1.03)</td>
<td>-0.12</td>
</tr>
<tr>
<td>Tassimo</td>
<td>0.56 (0.28)</td>
<td>1.23 (0.51)</td>
<td>0.91 (1.33)</td>
<td>5.49 (2.37)</td>
<td>-0.55</td>
</tr>
</tbody>
</table>

Notes. The summary statistics refer to the mean and standard deviation (in brackets) of the average daily budget, the number of paid-search keywords, the average daily budget per keyword, and the average position for each firm. The correlations that are significant at a 90% level are bolded.

As shown above, 10 out of 11 firms display a negative correlation between the average daily budget per keyword and average ad position, among which 7 are significant. This finding suggests that advertisers’ daily budgets can affect position outcomes.

A2. Survey Description
We executed an online survey to ask search advertisers about the likelihood of budget exhaustions. The survey link was posted on eight Internet forums devoted to search advertising. It was also advertised on Facebook, targeting users whose profiles match interests in “Pay per click, Search engine marketing, Keyword research or Search engine optimization.” In our survey design, we explicitly asked subjects about their search advertising experience and only recruited those who have used search advertising. We run this survey from Jan to April 2014. In total, we have 647 subjects entering the survey with 107 subjects who completed it, taking an average of 4 minutes each. The key question we are interested in is: “On average, how often do you use up your daily budget?” Table A2 indicates that less than 20% of respondents used up their budgets all the time, and more than 60% users’ budgets are not depleted on a weekly basis. These results indicate the heterogeneity in the rate of advertisers’ budget exhaustion.

Table A2. Survey Results on the Rate of Budget Exhaustion

<table>
<thead>
<tr>
<th>On average, how often do you use up your daily budget?</th>
<th>Response</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Never</td>
<td>32</td>
<td>30%</td>
</tr>
<tr>
<td>Several times within a year</td>
<td>18</td>
<td>17%</td>
</tr>
<tr>
<td>Once a month</td>
<td>15</td>
<td>14%</td>
</tr>
<tr>
<td>Several times within a month</td>
<td>10</td>
<td>9%</td>
</tr>
<tr>
<td>Once a week</td>
<td>6</td>
<td>6%</td>
</tr>
<tr>
<td>Several times within a week</td>
<td>6</td>
<td>6%</td>
</tr>
<tr>
<td>All the time</td>
<td>20</td>
<td>19%</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>107</td>
<td>100%</td>
</tr>
</tbody>
</table>

Appendix B. Definitions, Tables, and Figures for Equilibrium Characterization

We summarize all notations in Table B1, followed by a description of advertisers’ profit functions in Table B2. Then we present the UNE in the whole budget space in Table B3, in which each budget region is depicted in Figure B1. For the ease of exposition, we propose several handy definitions, which are repetitively used in the proof of propositions.

**Definition B1.** Advertiser $i$ has *jamming intention* when competing advertiser $j$’s budget is exhaustible under $b_j$ at the first position ($B_j < \sigma b_j$); advertiser $i$ has *jamming apprehension* if her own budget is exhaustible under competing advertiser $j$’s bid at the first position ($B_i < \sigma b_j$).

**Definition B2.** The UBRF refers to the *best-response function with an undominated bid.*

Table B1. Summary of Notations

<table>
<thead>
<tr>
<th>Notation</th>
<th>Expression</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_i$</td>
<td></td>
<td>Advertiser $i$’s value-per-click</td>
</tr>
<tr>
<td>$B_i$</td>
<td></td>
<td>Advertiser $i$’s budget</td>
</tr>
<tr>
<td>$\sigma$</td>
<td></td>
<td>Click ratio between the 1\textsuperscript{st} and the 2\textsuperscript{nd} position</td>
</tr>
<tr>
<td>$r$</td>
<td></td>
<td>Reservation value</td>
</tr>
</tbody>
</table>
Bid threshold s.t. \( i \) is indifferent between two positions with neither jamming intention nor apprehension.

\[
\tilde{b}_i = \frac{\pi_i B_i}{\pi_i + B_i - r}
\]

Bid threshold s.t. \( i \) is indifferent between two positions with no jamming intention but apprehension.

\[
\hat{b}_i = \frac{r + \sqrt{r^2 + \frac{(\sigma - 1)(\pi_i - r)B_i}{(1 - \frac{1}{\sigma})}}}{\sigma^2}
\]

Bid threshold s.t. \( i \) is indifferent between two positions with jamming intention but no apprehension.

\[
\tilde{b}_{ij} = \frac{\pi_i B_i + (1 - \frac{1}{\sigma})(\pi_i - r)B_j}{B_i + \sigma(\pi_i - r)}
\]

Bid threshold s.t. \( i \) is indifferent between two positions with both jamming intention and apprehension.

\[
\hat{b}_{ij} = \frac{(\sigma - 1)r B_j}{\sigma(\pi_i - B_i)}
\]

Bid threshold s.t. \( i \)'s budget is exactly inexhaustible by the reservation value given that \( i \) has jamming intention.

\[
\tilde{b}_{ij} = \frac{\sigma b_i}{\sigma b_i}
\]

Inexhaustible budget threshold under \( b_i \).

\[
\hat{b}_{ij} = \frac{\sigma \hat{b}_i}{\sigma \hat{b}_i}
\]

Inexhaustible budget threshold under \( \hat{b}_i \).

\[
\check{b}_{ij} = \frac{\sigma \check{b}_i}{\sigma \check{b}_i}
\]

Inexhaustible budget threshold under \( \check{b}_i \).

\[
\tilde{b}_{ij} = \frac{\sigma \tilde{b}_i}{\sigma \tilde{b}_i}
\]

Budget threshold s.t. \( \tilde{b}_i = \tilde{b}_{ij} \).

Notes: The subscript \( i \) or \( ij \) of each notation indicates whether the functional form depends only on advertiser \( i \)'s information or on information from both advertiser \( i \) and \( j \).

Table B2. Characterization of Advertisers’ Profits When \( b_i > b_j \)

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Description</th>
<th>Conditions on Budgets</th>
<th>Advertising Profits</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Both are inexhaustible</td>
<td>( t_i, t_j = 1 )</td>
<td>( B_i &gt; \sigma b_j ) \n( B_i &gt; r )</td>
<td>( \Pi_i = \sigma(\pi_i - B_i) \n\Pi_j = \pi_j - r )</td>
</tr>
<tr>
<td>2. Only ( B_i ) is exhaustible</td>
<td>( t_i &lt; 1 = t_j )</td>
<td>( B_i &lt; \sigma b_j ) \n( B_j \geq \frac{B_i r}{\sigma b_j} + \left(1 - \frac{B_i}{\sigma b_j}\right) \sigma r )</td>
<td>( \Pi_i = \frac{(\sigma b_i - B_i)}{b_j} \n\Pi_j = \frac{(\pi_j - r)B_i}{\sigma b_j} + \left(1 - \frac{B_i}{\sigma b_j}\right) \sigma (\pi_j - r) )</td>
</tr>
<tr>
<td>3. Only ( B_j ) is exhaustible</td>
<td>( t_j &lt; 1 = t_i )</td>
<td>( B_i \leq \sigma b_j ) \n( B_j &lt; r )</td>
<td>( \Pi_i = \frac{\sigma b_i - B_i}{r} \n\Pi_j = \frac{(\pi_j - r)B_j}{r} )</td>
</tr>
<tr>
<td>4. Both are exhaustible and ( i ) leaves sooner</td>
<td>( t_i \leq t_j \leq 1 )</td>
<td>( B_i \leq \sigma b_j ) \n( B_j \geq \frac{B_i r}{\sigma b_j} ) \n( \frac{B_i}{\sigma b_j} + \frac{B_j}{\sigma b_j} \leq 1 )</td>
<td>( \Pi_i = \frac{(\pi_i - b_j)B_i}{b_j} \n\Pi_j = \frac{(\pi_j - r)B_j}{r} )</td>
</tr>
<tr>
<td>5. Both are exhaustible and ( j ) leaves sooner</td>
<td>( t_j &lt; t_i \leq 1 )</td>
<td>( B_i &gt; \frac{\sigma b_j}{r} ) \n( B_j &lt; r )</td>
<td>( \Pi_i = \frac{B_j \sigma (\pi_i - b_j) r}{r} + \frac{B_i - \rho b_j}{r} (\pi_i - r) \n\Pi_j = \frac{(\pi_j - r)B_j}{r} )</td>
</tr>
</tbody>
</table>

Table B3. Characterization of the UNE

<table>
<thead>
<tr>
<th>Area No. in Fig. B1</th>
<th>1st Position</th>
<th>Strategy Type</th>
<th>Subcases</th>
<th>( b_i )</th>
<th>( b_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (where ( j ))</td>
<td>( \Theta_{Aj} )</td>
<td></td>
<td></td>
<td>( \tilde{b}_{ij} - \varepsilon )</td>
<td>( \tilde{b}_{ij} )</td>
</tr>
</tbody>
</table>
\[ \tilde{b}_{ji} > \tilde{b}_{ij} \] & 2
\hline
3 & 17
\midrule
j & \Theta_{Dj} & \tilde{b}_i & \forall b \in \left[ \tilde{b}_{ji}, \frac{B_i}{\sigma} \right] \\
\hline
1 \text{ (where } \tilde{b}_{ji} \leq \tilde{b}_{ij}) & 4 & 9
\midrule
i & \Theta_{Ai} & \tilde{b}_{ij} & \tilde{b}_{ij} - \varepsilon \\
\hline
5
\midrule
i & \Theta_{Ai} & \tilde{b}_i & \tilde{b}_i - \varepsilon \\
\hline
6
\midrule
i & \Theta_{BF} & \forall b \in \left[ \tilde{b}_i, \frac{B_i}{\sigma} \right] & b_j^* \\
\hline
7
\midrule
i & \Theta_{BF} & \forall b \in \left[ \tilde{b}_{ij}, \frac{B_i}{\sigma} \right] & b_j^* \\
\hline
8 & 14
\midrule
i & \Theta_{Di} & \forall b \in \left[ \tilde{b}_{ij}, \frac{B_i}{\sigma} \right] & \tilde{b}_j \\
\hline
10
\midrule
i \text{ or } j \text{ with equal prob} & \Theta_R & r & r \\
\hline
11
\midrule
i & \Theta_R & \text{if } B_i < \sigma r & \forall b \in \left[ \tilde{b}_{ij}, \tilde{b}_{ij} \right] & r \\
\midrule
i & \Theta_R & \text{if } B_i \geq \sigma r & \forall b \in \left[ \tilde{b}_{ij}, +\infty \right) & r \\
\hline
12
\midrule
i & \Theta_R & \forall b \in \left[ \tilde{b}_{ij}, +\infty \right) & r \\
\hline
13 & 15 \text{ where } \tilde{b}_{ji} \leq \tilde{b}_{ij}
\midrule
i & \Theta_{Ai} & \text{if } \tilde{b}_{ji} \geq \tilde{b}_{ij} & \tilde{b}_{ij} & \tilde{b}_{ij} - \varepsilon \\
\midrule
i & \Theta_{Pi} & \text{if } \tilde{b}_{ji} < \tilde{b}_{ij} & \tilde{b}_{ij} & \tilde{b}_{ij} \\
\hline
16 & 15 \text{ where } \tilde{b}_{ji} > \tilde{b}_{ij}
\midrule
j & \Theta_{A} & \text{if } \tilde{b}_{ij} \geq \tilde{b}_{ji} & \tilde{b}_{ji} & \tilde{b}_{ji} - \varepsilon \\
\midrule
j & \Theta_{Pj} & \text{if } \tilde{b}_{ij} < \tilde{b}_{ji} & \tilde{b}_{ij} & \tilde{b}_{ij} \\
\hline
18
\midrule
j & \Theta_R & \text{if } B_j < \sigma r & \forall b \in \left[ \tilde{b}_{ji}, \tilde{b}_{ji} \right] & r \\
\midrule
j & \Theta_R & \text{if } B_j \geq \sigma r & \forall b \in \left[ \tilde{b}_{ji}, +\infty \right) & r \\
\hline
19
\midrule
j & \Theta_R & r & \forall b \in \left[ \tilde{b}_{ji}, +\infty \right) \\
\end{array}
Appendix C. Equilibrium Derivation When $r < \min\left\{\frac{B_i}{\sigma}, \frac{B_j}{\sigma}\right\}$

Outline of the proof of Proposition 1a–1c. We derive the equilibrium bids in the whole budget space in several steps. First, we characterize the best-response functions for both advertisers based on whether or not they have jamming intention/apprehension (Lemma C1). Second, we separate the whole budget space into smaller areas based on the ordinal relationship between the budget thresholds under which advertisers are indifferent between two ad positions (Lemma C2). Third, we characterize the UNE in the half budget space where $B_j \geq B_i$ (Lemma C3). Fourth, we show that some ambiguous conditions in Lemma C3 can be simplified when we extend the UNE to the whole budget space (Lemma C4 and C5). Finally, we derive the UNE in the whole budget space and prove all arguments made in Proposition 1a–1c. Here we do not assume $\pi_i \geq \pi_j$ in Lemma C1–C3.

**Lemma C1.** The best-response function of advertiser i’s bid is as follows.

1. When advertiser i has neither jamming intention nor jamming apprehension ($b_j \leq \frac{B_i}{\sigma}$ & $b_j \leq \frac{B_j}{\sigma}$), her best-response function is $b_i(b_j) = \begin{cases} x | x > b_j & \text{if } b_j < b_i^* \\ x | x \leq b_j & \text{if } b_j \geq b_i^* \end{cases}$

2. When advertiser i has jamming apprehension but no jamming intention ($\frac{B_i}{\sigma} \leq b_j < \frac{B_j}{\sigma}$), her best-response function is $b_i(b_j) = \begin{cases} x | x > b_j & \text{if } b_j < \tilde{b}_i \\ x | x \leq b_j & \text{if } b_j \geq \tilde{b}_i \end{cases}$. 
3. When advertiser \( i \) has jamming intention but no jamming apprehension \( (\frac{B_i}{\sigma} \leq b_j < \frac{B_i}{\sigma}) \), her best-response function is \( b_i(b_j) = \begin{cases} x | x > b_j & \text{if } b_j < \bar{b}_{ij} \\ b_j - \epsilon & \text{if } b_j \geq \bar{b}_{ij} \end{cases} \).

4. When advertiser \( i \) has both jamming intention and jamming apprehension \( (\frac{B_i}{\sigma} \leq b_j \leq \frac{B_i}{\sigma} \leq b_j) \), her best-response function is \( b_i(b_j) = \begin{cases} x | x > b_j & \text{if } b_j < \bar{b}_{ij} \\ b_j - \epsilon & \text{if } b_j \geq \bar{b}_{ij} \end{cases} \).

**Proof of Lemma C1.** When \( r < \min \left\{ \frac{B_i}{\sigma} + \frac{B_j}{\sigma} \right\} \), advertisers’ profit functions can be simplified as

\[
\Pi_i(b_i, b_j) = \begin{cases} 
\sigma(\pi_i - b_j) \min\left\{ \frac{B_i}{\sigma b_j}, 1 \right\} & \text{if } b_i > b_j \\
(\pi_i - r) \min\left\{ \frac{B_j}{\sigma b_i}, 1 \right\} + \sigma(\pi_i - r) \max\left\{ 0, 1 - \frac{B_i}{\sigma b_i} \right\} & \text{if } b_i < b_j 
\end{cases}
\]

We next prove Lemma C1 case by case. In case 1 where \( b_j \leq \frac{B_i}{\sigma} \leq b_j \), advertiser \( i \)’s profit is \( \Pi_i(b_i, b_j) = \begin{cases} \sigma(\pi_i - b_j) & \text{if } b_i > b_j \\
\pi_i - r & \text{if } b_i < b_j \end{cases} \).

Because \( \sigma(\pi_i - b_j) \geq \pi_i - r \) is equivalent to \( b_j \leq b_i^* \), advertiser \( i \)’s best-response function is \( b_i(b_j) \) described in Lemma C1. In case 2 where \( \frac{B_j}{\sigma} \leq b_j \leq \frac{B_i}{\sigma} \), \( \Pi_i(b_i, b_j) = \begin{cases} \frac{\pi B_i}{b_j} - b_i & \text{if } b_i > b_j \\
\pi_i - r & \text{if } b_i < b_j \end{cases} \).

Because \( \frac{\pi B_i}{b_j} - b_i \geq \pi_i - r \) is equivalent to \( b_j \leq b_{ij}^* \), advertiser \( i \)’s best-response function is \( b_i(b_j) \) described in Lemma C1. The best-response functions in cases 3 and 4 are derived in a similar way. \( \square \)

**Lemma C2.** Four sets of inequality equivalences hold among four budget thresholds.

1. \( B_i \leq B_i^* \leftrightarrow B_i \leq \bar{B}_i \leftrightarrow \bar{B}_i \leq B_i^* \).
2. \( B_i \leq \bar{B}_{ij} \leftrightarrow B_i \leq \bar{B}_{ij} \leftrightarrow \bar{B}_{ij} \leq B_i\).
3. \( B_j \leq B_j^* \leftrightarrow B_j \leq \bar{B}_{ij} \leftrightarrow \bar{B}_{ij} \leq B_j^* \).
4. \( B_j \leq \bar{B}_i \leftrightarrow B_j \leq \bar{B}_i \leftrightarrow \bar{B}_i \leq \bar{B}_i\).

where \( B_i^* = \sigma b_i^*, \bar{B}_i = \sigma \bar{b}_i, \bar{B}_{ij} = \sigma b_{ij}, \bar{B}_{ij} = \sigma \bar{b}_{ij} \).

**Proof of Lemma C2.** For the first set of inequalities, \( B_i \leq B_i^* \leftrightarrow B_i \leq (\sigma - 1)\pi_i + r \leftrightarrow B_i \leq \frac{\sigma \pi B_i}{\pi_i + B_i - r} \leftrightarrow B_i \leq \bar{B}_i \leftrightarrow \sigma \pi B_i \leq (\sigma - 1)\pi_i + r \leftrightarrow \bar{B}_i \leq B_i^* \). Other inequality equivalences can be verified in a similar way. \( \square \)

Lemma C2 implies that \( \bar{B}_i \) always lies between \( B_i \) and \( B_i^* \), \( \bar{B}_{ij} \) lies between \( B_i \) and \( \bar{B}_{ij} \), \( \bar{B}_{ij} \) lies between \( B_j \) and \( B_j^* \), and \( \bar{B}_{ij} \) lies between \( B_j \) and \( \bar{B}_i \). The intuition behind these inequalities is that when both advertisers’ budgets are insufficient, the bid threshold under which an advertiser is indifferent between two positions is always higher when the advertiser does not have jamming apprehension. This explains the first two sets of inequality equivalence as \( \bar{B}_i \) and \( \bar{B}_{ij} \) are budget thresholds for advertisers who have jamming apprehension whereas \( B_i^* \) and \( \bar{B}_{ij} \) are budget thresholds for advertisers who do not. Similarly, the bid threshold under which an advertiser is indifferent between two positions is always lower when the advertiser has jamming intention and this explains the last two sets of inequality equivalences.

Next we characterize the UNE in the half-budget space where \( B_j \geq B_i \). The ordinal relationships among budget thresholds in Lemma C2 sheds light on how to separate the half-budget space into sub-
areas. In particular, we first specify the range of \( B_i \) by ordering \( \{B_i, \tilde{B}_i, \tilde{B}_{ij}, B_i^*, B_j^*\} \) into five cases and then specify the range of \( B_j \) by ordering \( \{B_j, \tilde{B}_j, \tilde{B}_{ji}\} \) within each case. Cases 1–5 below correspond to the increase in \( B_i \) from zero to infinite: 1. \( B_i \leq \tilde{B}_i \leq \tilde{B}_{ij} \leq \min\{B_i^*, B_j^*\} \); 2. \( B_i \leq \tilde{B}_{ij} \leq \tilde{B}_i \leq \min\{B_i^*, B_j^*\} \); 3. \( B_i^* \leq B_i \leq \tilde{B}_{ij} \leq B_j^* \); 4. \( B_j^* \leq \tilde{B}_{ji} \leq B_i \leq \tilde{B}_i \leq B_j^* \); 5. \( \max\{B_i^*, B_j^*\} \leq B_i \). Within each case, we further consider several subcases by comparing \( B_j \) with \( \{\tilde{B}_i, \tilde{B}_{ji}\} \). We demonstrate the derivation of UNE in one case in the following example. The UNE in other cases can be derived in a similar way.

**Example.** Equilibrium derivation when \( B_i \leq B_j \leq \tilde{B}_i \leq \tilde{B}_{ji} \leq \min\{B_i^*, B_j^*\} \).

**Step 1:** Characterize the best-response functions of both advertisers as

\[
b_i(b_j) = \begin{cases}
  x | x > b_j & \text{if } b_j < \tilde{b}_{ij} \\
  b_j - \epsilon & \text{if } b_j \geq \tilde{b}_{ij}
\end{cases}
\]

\[
b_j(b_i) = \begin{cases}
  x | x > b_i & \text{if } b_i < \tilde{b}_{ij} \\
  b_i - \epsilon & \text{if } b_i \geq \tilde{b}_{ij}
\end{cases}
\]

**Step 2:** Eliminate all weakly dominated bids.

For advertiser \( i \), it is clear that any bid below \( \tilde{b}_{ij} \) is weakly dominated by \( \tilde{b}_{ij} \) because by bidding below \( \tilde{b}_{ij} \), advertiser \( i \) obtains the same or less profit than bidding \( \tilde{b}_{ij} \) if advertiser \( j \) bids below \( \tilde{b}_{ij} \), and advertiser \( i \) obtains strictly less profit if advertiser \( j \) bids above \( \tilde{b}_{ij} \). Similarly, any bid below \( \tilde{b}_{ij} \) is weakly dominated by \( \tilde{b}_{ij} \) for advertiser \( j \) here. Thus, the best-response functions with undominated bids, or the UBRFs are

\[
b_i(b_j) = \begin{cases}
  x | x > b_j & \text{if } b_j < \tilde{b}_{ij} \\
  b_j - \epsilon & \text{if } \tilde{b}_{ij} \leq b_j
\end{cases}
\]

\[
b_j(b_i) = \begin{cases}
  x | x > \tilde{b}_{ij} & \text{if } b_i < \tilde{b}_{ij} \\
  b_i - \epsilon & \text{if } b_i \geq \tilde{b}_{ij}
\end{cases}
\]

**Step 3:** The UNE is determined by the intersection of the best-response functions.

As shown in Figure A1, the equilibrium bid in this case is

\[
(b_i, b_j) = \begin{cases}
  (\tilde{b}_{ij}, \tilde{b}_{ij} - \epsilon) & \text{if } \tilde{b}_{ij} \geq \tilde{b}_{ji} \\
  (\tilde{b}_{ji} - \epsilon, \tilde{b}_{ji}) & \text{if } \tilde{b}_{ij} < \tilde{b}_{ji}
\end{cases}
\]

\[14\] There are actually two intersections of advertisers’ best-response functions in Figure C1. However, we treat two equilibria \((b, b + \epsilon)\) and \((b - \epsilon, b)\) as the same because the equilibrium outcome for all parties are the same given that \( \epsilon \to 0 \). We impose this equilibria equivalence in the rest of the paper.

---

\[13\] Lemma C2 suggests that the order of \( \{B_i, B_j, B_i^*, B_j^*\} \) in the half-budget space where \( B_j \geq B_i \) can only have six possibilities: 1. \( B_i \leq B_j \leq \tilde{B}_{ij} \leq \tilde{B}_i \leq B_i^* \); 2. \( B_i \leq \tilde{B}_i \leq \tilde{B}_{ij} \leq B_j \leq B_i^* \); 3. \( B_i \leq \tilde{B}_{ij} \leq \tilde{B}_i \leq B_i^* \leq B_j \); 4. \( B_i \leq \tilde{B}_i \leq \tilde{B}_{ij} \leq B_i^* \leq B_j \); 5. \( \tilde{B}_i \leq \tilde{B}_{ij} \leq B_i^* \leq B_j \); 6. \( B_i^* \leq \tilde{B}_i \leq \tilde{B}_{ij} \leq B_i \). Similarly, for advertiser \( j \), the order of \( \{B_i, B_j, B_i^*, B_j^*\} \) also only have six possibilities: 1. \( B_j \leq B_i \leq \tilde{B}_{ji} \leq \tilde{B}_j \leq B_j^* \); 2. \( B_j \leq \tilde{B}_j \leq \tilde{B}_{ji} \leq B_i \leq B_j^* \); 3. \( B_j \leq \tilde{B}_{ji} \leq B_j^* \leq B_i \); 4. \( B_j \leq \tilde{B}_{ji} \leq B_j^* \leq B_i \); 5. \( B_j^* \leq \tilde{B}_j \leq \tilde{B}_{ji} \leq B_i \); 6. \( B_j^* \leq \tilde{B}_j \leq \tilde{B}_{ji} \leq B_j \).
Figure C1. The UNE When $B_i \leq B_j \leq \bar{B}_i \leq \bar{B}_{ji} \leq \min\{B_i^*, B_j^*\}$

Notes. We assume $\tilde{b}_{ij} < \tilde{b}_{ji}$ in Figure C1. The red (green) part stands for the UBRF of advertiser $i(j)$. The UNE is determined by the intersections of two parts.

**Lemma C3.** The UNE in the half-budget space where $B_j \geq B_i$ is described in Table C1.

Table C1. The UNE When $B_j \geq B_i$  

<table>
<thead>
<tr>
<th>Case</th>
<th>Description</th>
<th>Conditions</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>$B_j \leq B_i \leq \bar{B}<em>i \leq \bar{B}</em>{ji} \leq \min{B_i^<em>, B_j^</em>}$</td>
<td>$b_i, b_j = \tilde{b}<em>{ij}, \tilde{b}</em>{ij} - \epsilon$ or $\bar{b}<em>{ji} - \epsilon, \tilde{b}</em>{ij} - \epsilon, \bar{b}<em>{ji} - \epsilon, \tilde{b}</em>{ij}, \forall b \in \left[\bar{b}_{ji}, \frac{B_i}{\sigma}\right]$</td>
<td>$b_i, b_j = \tilde{b}<em>{ij}, \tilde{b}</em>{ij} - \epsilon$ or $\bar{b}<em>{ji} - \epsilon, \tilde{b}</em>{ij} - \epsilon, \bar{b}<em>{ji} - \epsilon, \tilde{b}</em>{ij}, \forall b \in \left[\bar{b}_{ji}, \frac{B_i}{\sigma}\right]$</td>
</tr>
<tr>
<td>Case 2</td>
<td>$B_j \leq \bar{B}<em>i \leq B_i \leq \bar{B}</em>{ji} \leq \min{B_i^<em>, B_j^</em>}$</td>
<td>$b_i, b_j = \tilde{b}<em>{ij}, \tilde{b}</em>{ij} - \epsilon$ or $\bar{b}<em>{ji} - \epsilon, \tilde{b}</em>{ij} - \epsilon, \bar{b}<em>{ji} - \epsilon, \tilde{b}</em>{ij}, \forall b \in \left[\bar{b}_{ji}, \frac{B_i}{\sigma}\right]$</td>
<td>$b_i, b_j = \tilde{b}<em>{ij}, \tilde{b}</em>{ij} - \epsilon$ or $\bar{b}<em>{ji} - \epsilon, \tilde{b}</em>{ij} - \epsilon, \bar{b}<em>{ji} - \epsilon, \tilde{b}</em>{ij}, \forall b \in \left[\bar{b}_{ji}, \frac{B_i}{\sigma}\right]$</td>
</tr>
<tr>
<td>Case 3</td>
<td>$B_i^* \leq \bar{B}<em>i \leq B_i \leq \bar{B}</em>{ji} \leq B_i^*$</td>
<td>$b_i, b_j = \tilde{b}<em>{ij} - \epsilon, \tilde{b}</em>{ij}$</td>
<td>$b_i^*, \forall b \in \left[\tilde{b}_{ji}, \frac{B_i}{\sigma}\right]$</td>
</tr>
<tr>
<td>Case 4</td>
<td>$B_j^* \leq \bar{B}<em>i \leq B_i \leq \bar{B}</em>{ji} \leq B_i^*$</td>
<td>$b_i, b_j = \tilde{b}<em>{ij}, \tilde{b}</em>{ij} - \epsilon, \bar{b}<em>{ji}, \bar{b}</em>{ji} - \epsilon, \bar{b}<em>{ji}, \forall b \in \left[\bar{b}</em>{ji}, \frac{B_i}{\sigma}\right]$</td>
<td>$b_i^*, \forall b \in \left[\tilde{b}_{ji}, \frac{B_i}{\sigma}\right]$</td>
</tr>
<tr>
<td>Case 5</td>
<td>$\max{B_i^<em>, B_j^</em>} \leq B_i$</td>
<td>$b_i^*, \forall b \in \left[\frac{B_i}{\sigma}\right]$ if $\pi_i \leq \pi_j$</td>
<td>$b_j^*, \forall b \in \left[\frac{B_j}{\sigma}\right]$ if $\pi_i &gt; \pi_j$</td>
</tr>
</tbody>
</table>
Proof of Lemma C3. We prove the UNE in Case 1 for a demonstration. In Case 1 where \( B_i \leq \hat{B}_{ij} \leq \bar{B}_{ij} \leq \min\{B_i', B_j'\} \), the subcase 1 where \( B_j \in (B_i, \bar{B}_i) \) has been discussed in the example above. In subcase 2 where \( B_j \in [\hat{B}_i, \bar{B}_{ji}] \), the UBRF for advertisers \( i \) is \( b_i(b_j) = \begin{cases} \hat{b}_i & \text{if } \hat{b}_i < b_j \leq \bar{b}_i, \\ b_j - \epsilon & \text{if } \frac{B_j}{\sigma} < b_j, \end{cases} \). To see this, we prove that any bid \( b \in (r, \hat{b}_i) \cup \left( \hat{b}_i, \frac{B_j}{\sigma} \right) \) is weakly dominated by \( \hat{b}_i \) for advertiser \( i \). It can be verified that for any \( b \in (r, \hat{b}_i) \), advertiser \( i \) obtains the same profit under \( b \) and \( \hat{b}_i \) when \( b \leq \hat{b}_i \) or \( b \geq \bar{b}_i \) but strictly higher profit under \( \hat{b}_i \) when \( b \in (\hat{b}_i, \bar{b}_i) \). Similarly, for any \( b \in \left( \hat{b}_i, \frac{B_j}{\sigma} \right) \), advertiser \( i \) obtains the same profit under \( b \) and \( \hat{b}_i \) when \( b \leq \hat{b}_i \) or \( b \geq \bar{b}_i \) but strictly higher profit under \( \hat{b}_i \) when \( b \in (\hat{b}_i, \bar{b}_i) \). Meanwhile, the UBRF for advertiser \( j \) is \( b_j(b_i) = \begin{cases} x|x > \hat{b}_{ji} & \text{if } b_i < \hat{b}_{ji}, \\ b_i - \epsilon & \text{if } \hat{b}_{ji} \leq b_i. \end{cases} \). Based on Lemma C2, \( \bar{b}_{ji} \) lies between \( \bar{b}_{ji} \) and \( \frac{B_j}{\sigma} \). Thus, the UNE in this subcase 2 is \( (\bar{b}_{ji} - \epsilon, \bar{b}_{ji}) \), under which advertiser \( j \) is indifferent between two positions, whereas advertiser \( i \) strictly prefers to jam at the second position. In subcase 3 where \( B_j \in [\bar{B}_{ji}, +\infty) \), the UBRF for advertiser \( i \) is the same as in subcase 2, while the UBRF for advertiser \( j \) becomes \( b_j(b_i) = \begin{cases} x|x > \bar{b}_{ji} & \text{if } b_i < \bar{b}_{ji}, \\ b_i - \epsilon & \text{if } \bar{b}_{ji} \leq b_i. \end{cases} \). Because \( \hat{b}_i \leq \bar{b}_{ji} \leq \frac{B_j}{\sigma} \), in this subcase, the UNE is \( (\hat{b}_i, \forall b \in \left[ \bar{b}_{ji}, \frac{B_j}{\sigma} \right]) \). The proof of UNE in remaining cases is similar and thereby neglected. □

So far we have derived the UNE in the half-budget space where \( B_j \geq B_i \). However, the budget area specified by \( \hat{B}_i \leq \bar{B}_{ij} \) has not been well characterized and the order of \( \bar{b}_{ji} \) and \( \bar{b}_{ij} \) is also uncertain when the difference between advertisers’ budgets is small. Therefore, we propose the following two lemmas to provide a more deterministic characterization of the UNE in the whole budget space.

Lemma C4. Suppose advertiser \( i \) has a higher value-per-click and both advertisers’ budget are insufficient, then we have:

1. \( \hat{B}_j \leq \bar{B}_{ij} \) always holds; 2. \( \hat{B}_i \leq \bar{B}_{ij} \) is equivalent to \( B_i \leq \bar{B}_{ij} \), where \( B_{ij} = \frac{\sigma \pi_i (\pi_i - \sigma r)}{2(\sigma - 1)(\pi_j - r)} \left( 1 - \sqrt{1 - \frac{4(\sigma - 1)(\pi_i - r)(\pi_j - r)}{\sigma(\pi_i - \sigma r)^2}} \right) - \pi_i + r. \)

Proof of Lemma C4. We intend to prove two statements here. First, we show that \( \hat{B}_j \leq \bar{B}_{ij} \) holds when \( B_j \in (r, B_j') \). Second, we prove that \( \hat{B}_i \) and \( \bar{B}_{ji} \) intersect only once at \( B_i = \bar{B}_{ij} \) when \( B_i \in (r, B_i') \).

We start with the proof of the first statement. Note that \( \bar{B}_{ij} \geq \hat{B}_i \) is equivalent to \( f(x) \geq 0 \), where \( f(x) = (\pi_i - r)(\sigma - 1)x^2 + \sigma \pi_i (\sigma r - \pi_i)x + (\pi_j - r)\sigma^2 \pi_i \) and \( x = \pi_j + B_j - r \). We can see that \( f(x) \) is a convex function. When \( B_j \) stays at the two boundaries of the interval \( (r, B_j') \), we have \( f(\pi_j) = (\sigma - 1)\pi_j^2 \pi_i > 0 \) and \( f(\sigma \pi_j) = \sigma(\sigma - 1)\pi_j^2 (\sigma \pi_i - \pi_j + r) > 0 \). Next we show that \( f(x) \geq 0 \) for any \( x \in (\pi_j, \sigma \pi_j) \) by considering three cases.

Case 1: \( \pi_j \leq \sigma r \). In this case, suppose there exists an \( x \in (\pi_j, \sigma \pi_j) \), s.t. \( f(x) < 0 \). Since \( f(\pi_j) > 0 \) and \( f(\sigma \pi_j) > 0 \), the two zero points of \( f(x) \), denoted by \( x_1 \) and \( x_2 \), must be located within the interval.
Thus, \( x_1 + x_2 \) should be positive. However, since \( x_1 + x_2 = \frac{(\pi_j - \sigma\pi_j)}{\pi_i - \pi_r} < 0 \), this contradiction implies \( f(x) \geq 0 \) when \( x \in (\pi_j, \pi_r) \).

Case 2: \( \pi_j > \sigma r \) \& \( \sigma > 2 \). Following the previous logic, if three exists an \( x \in (\pi_j, \pi_r) \), s.t. \( f(x) < 0 \), we should have \( x_1 + x_2 = \frac{(\pi_j - \sigma\pi_j)}{\pi_i - \pi_r} > 2\pi_j \), which is equivalent to \( \sigma(\pi_j - \sigma r) > 2(\sigma - 1)(\pi_i - r) \). Because \( \sigma > 2 \) \& \( \pi_i \geq \pi_j \), the reverse is true. This contradiction implies \( f(x) \geq 0 \) when \( x \in (\pi_j, \pi_r) \).

Case 3: \( \pi_j > \sigma r \) \& \( \sigma \leq 2 \). In this case, we prove that the two real-valued zero points of \( f(x) \) must satisfy \( x_1 + x_2 = 2\pi_j \), which implies that \( f(x) \geq 0 \) when \( x \in (\pi_j, \pi_r) \). To see this, we first note that the necessary and sufficient condition for \( f(x) \) to have real-valued zero points is \( \pi_i - r \leq \frac{\sigma(\pi_j - \sigma r)^2}{4(\sigma - 1)(\pi_i - r)} \). Thus, \( x_1 + x_2 = \frac{(\pi_j - \sigma\pi_j)}{\pi_i - \pi_r} \geq \frac{4\pi_j(\pi_j - r)}{\pi_i - \sigma r} > 2\pi_j \), the last inequality of which is equivalent to \( 2(\pi_j - r) > \sigma(\pi_j - \sigma r) \).

For the second statement, note that \( \bar{B}_i(B_i = r) > r \) is \( \bar{B}_i(B_i = r) \) and \( \bar{B}_i(B_i = B'_i) < B'_i \) is \( \bar{B}_i(B_i = B'_i) \), so there is at least one intersection of \( \bar{B}_i \) and \( \bar{B}_j \) when \( B_i \in (r, B'_i) \). Based on the proof above, there are at most two intersections of \( \bar{B}_i \) and \( \bar{B}_j \) for \( B_i \in (-\infty, +\infty) \). Since it is impossible for both of these two intersections to locate within \( (r, B'_i) \) (otherwise the ordinal relationship between \( \bar{B}_i \) and \( \bar{B}_j \) should be the same at two boundaries), we prove that there is only one intersection of \( \bar{B}_i \) and \( \bar{B}_j \) within this interval. Using the same logic, we can further prove that this intersection must be within \( B_i \in (r, B'_i) \) because \( \bar{B}_i(B_i = B'_i) = B'_i \) when \( B_i \in \bar{B}_i \).

**Lemma C5.** Suppose advertiser \( i \) has both a higher value-per-click and a larger budget, when the budget difference is small \( (B_j \leq B_i \leq \bar{B}_j) \) we have \( \bar{B}_i \geq \bar{B}_j \).

**Proof of Lemma C5.** We want to show that given any \( B_j \), \( \bar{B}_i(B_i) \geq \bar{B}_j(B_i) \) when \( B_i \in [B_j, \bar{B}_j] \). First, we prove that it is true at the two boundaries. When \( B_i = B_j \), \( \bar{B}_i(B_i) = \frac{b_i}{B_j + \sigma(\pi_i - r)} \). Its derivative w.r.t. \( \pi_i \) is \( \frac{\partial \bar{B}_i}{\partial \pi_i} = \frac{\sigma(\pi_j - \sigma r)^2}{(2 - \frac{1}{\sigma})B_j - \sigma r} > 0 \) under our assumption that \( B_j > \sigma r \). Thus, \( \bar{B}_i(B_i) > \bar{B}_j(B_i) \) when \( B_i = B_j \). When \( B_i = \bar{B}_j \), from Lemma C2, we have \( \bar{B}_i(B_i) = B_i < \bar{B}_j(B_i) \) because \( \bar{B}_j \leq \bar{B}_i \) from Lemma C4.

Next we show that \( \bar{B}_i(B_i) > \bar{B}_j(B_i) \) within the interval. Note that \( \bar{B}_i(B_i) > \bar{B}_j(B_i) \) is equivalent to \( f(B_j) > 0 \), where \( f(B_j) = [B_i\pi_i + (\pi_i - r)(1 - \frac{1}{\sigma})B_j]B_j + \sigma(\pi_j - r) = [B_j\pi_j + (\pi_j - r)(1 - \frac{1}{\sigma})B_i][B_i + \sigma(\pi_i - r)] \), which is a concave function of \( B_i \). Since \( f(B_j) \) and \( f(\bar{B}_j) \) are positive, we have \( f(\theta B_j + (1 - \theta)\bar{B}_j) > \theta f(B_j) + (1 - \theta)f(\bar{B}_j) > 0 \), for any \( \theta \in (0, 1) \). Thus, we have proved \( \bar{B}_i \geq \bar{B}_j \) when \( B_j \leq B_i \leq \bar{B}_j \).

Based on Lemma C3–C5, we are able to characterize the UNE in the budget space where \( r < \min \{ \frac{\bar{B}_1}{\sigma}, \frac{\bar{B}_2}{\sigma} \} \), assuming \( \pi_i \geq \pi_j \). First, Lemma C5 suggests that the UNE in the first subcase of case 2 described in Table C1 should be \( (b_i, b_j) = (\bar{B}_i, \bar{B}_j - \epsilon) \). Second, Lemma C4 suggests that we do not
need to consider the corresponding case 2 described in Table C1 in the other half-budget space where \( B_i \geq B_j \). These two observations help us extend the UNE derived in Lemma C3 to the whole budget space including regions 1 to 9 shown in Figure B1(a). The exact UNE in each region is reported in Table B3. The mathematical expression for each budget area in Figure B1(a) is given below. Area 1 corresponds to where \( B_j \in \left[B_i, \min\{\hat{B}_{ij}, \hat{B}_i\}\right] \). Area 2 corresponds to where \( B_i \leq \hat{B}_{ij} \& B_j \in \left[\hat{B}_i, \hat{B}_{ij}\right] \). Area 3 corresponds to where \( B_i < \hat{B}_{ij} \& B_j \geq \hat{B}_{ij} \). Area 4 corresponds to where \( B_i \in \left[\hat{B}_{ij}, \hat{B}_j^*\right] \& B_j \in \left[\max\{B_i, \hat{B}_{ij}\}, \hat{B}_i\right] \). Area 5 corresponds to where \( B_i \in \left[\hat{B}_{ij}, B_i^*\right] \& B_j > \hat{B}_i \). Area 6 corresponds to where \( B_i, B_j \geq B_i^* \). Area 7 corresponds to where \( B_i \geq \hat{B}_{ij} \& B_j \in \left[B_j^*, B_i^*\right] \). Area 8 corresponds to where \( B_i \geq \hat{B}_{ij} \& B_j < B_j^* \). Area 9 corresponds to where \( B_i \in \left[B_j, \hat{B}_{ij}\right] \).

Proof of Propositions 1a–1c. The analysis above indicates the following expressions for budget regions of different bid strategies: \( \Theta_{BF} = \{Area \ 6 \ & 7\} \), \( \Theta_{Al} = \{Area \ 4, 5, 9 \ & 1\ \ where \ \hat{b}_{ij} > \hat{b}_{ij}\} \), \( \Theta_{Aj} = \{Area \ 2 \ & 1 \ \ where \ \hat{b}_{ij} \leq \hat{b}_{ij}\} \), \( \Theta_{Di} = \{Area \ 8\} \) and \( \Theta_{Dj} = \{Area \ 3\} \). Statements regarding the conditions for advertisers’ uses of bid jamming and jamming protection are direct from the UNE described in Table B3.  □

Appendix D. Equilibrium Derivation When \( r \geq \min\left\{\frac{B_i}{\sigma}, \frac{B_j}{\sigma}\right\} \)

We first present several useful lemmas.

Lemma D1. When \( B_i \leq r \), the UBRF of advertiser i is \( b_i(b_j) = r \).

Proof of Lemma D1. When \( B_i \leq r \), advertiser i’s profit function is

\[
\Pi_i(b_i, b_j) = \begin{cases} 
(\pi_i - b_j) \frac{B_i}{b_j} & \text{if } b_i > b_j \\
(\pi_i - r) \frac{B_i}{r} & \text{if } b_i < b_j 
\end{cases}
\]

Since advertiser i’s budget will definitely be exhausted regardless of her position, her weakly undominated bid is to bid the reservation value \( r \). □

Lemma D2. \( \hat{b}_{ij} \geq r \leftrightarrow \hat{b}_{ij} \geq r \leftrightarrow \hat{b}_{ij} \geq \hat{b}_{ij} \leftrightarrow B_j \geq \frac{(\sigma - r)B_j}{\sigma - 1} \), where \( \hat{b}_{ij} = \frac{(\sigma - 1)rB_j}{\sigma (\sigma - B_i)} \).

Proof of Lemma D2. \( \hat{b}_{ij} \geq r \leftrightarrow \frac{(\sigma - 1)rB_j}{\sigma (\sigma - B_i)} \geq r \leftrightarrow B_j \geq \frac{(\sigma - B_i)}{\sigma - 1} \). Other parts of equivalence among these inequalities can be verified manually. □

Lemma D3. When \( r < B_i \leq \sigma r \), the UBRF \( b_i(b_j) \) is described below based on the range of \( B_j \):

a) When \( B_j < \frac{\sigma (\sigma - B_i)}{\sigma - 1} \), \( b_i(b_j) = r \):

\[
b_i(b_j) = \begin{cases} 
x | x \geq \hat{b}_{ij} \ & \text{if} \ & b_j < \hat{b}_{ij} \\
b_j - \epsilon \ & \text{if} \ & \hat{b}_{ij} \leq b_j \leq \hat{b}_{ij}; \\
\hat{b}_{ij} \ & \text{if} \ & b_j > \hat{b}_{ij}
\end{cases}
\]

b) When \( \frac{\sigma (\sigma - B_i)}{\sigma - 1} \leq B_j < \hat{B}_i \), \( b_i(b_j) = \begin{cases} 
x | x \geq \hat{b}_i \ & \text{if} \ & b_j < \hat{b}_i \\
\hat{b}_i \ & \text{if} \ & \hat{b}_i \leq b_j \leq \frac{B_j}{\sigma} \\
b_j - \epsilon \ & \text{if} \ & \frac{B_j}{\sigma} < b_j < b_{ij} \\
\hat{b}_{ij} \ & \text{if} \ & b_j > b_{ij}
\end{cases}
\]

c) When \( B_j \geq \hat{B}_i \), \( b_i(b_j) = \begin{cases} 
x | x \geq \frac{B_j}{\sigma} \ & \text{if} \ & b_j < \frac{B_j}{\sigma} \\
\frac{B_j}{\sigma} \ & \text{if} \ & \frac{B_j}{\sigma} \leq b_j \leq b_{ij} \\
b_j - \epsilon \ & \text{if} \ & b_j > b_{ij}
\end{cases}
\]
Proof of Lemma D3. We first derive the profit function of advertiser $i$, that is, $\Pi_i(b_i,b_j) = \begin{cases} (\pi_i - b_j) \frac{B_i}{b_j} & \text{if } b_i > b_j \\ (\pi_i - r) \min\{t_1, 1\} + \sigma(\pi_i - r) \max\{0, \min\{1-t_1, t_2\}\} & \text{if } b_i < b_j \end{cases}$, where $t_1 = \frac{B_j}{\sigma b_i}$ is the duration of advertiser $j$ in the auction. After advertiser $j$ drops out from the auction, $t_2 = \frac{B_i - t_1 r}{\sigma r}$ is the maximum remaining time for advertiser $i$ to stay at the first position before her own budget is exhausted. The profit function at the second position can be rewritten as follows:

$$
\Pi_i(b_i,b_j|b_i < b_j) = \begin{cases} 
\pi_i - r & \text{if } b_j \leq \frac{B_i}{\sigma} \\
(\pi_i - r) \frac{B_i}{\sigma b_i} + \sigma(\pi_i - r) \left(1 - \frac{B_i}{\sigma b_i} \right) & \text{if } \frac{B_i}{\sigma} < b_j \leq \hat{b}_{ij} \\
(\pi_i - r) \frac{B_i}{r} & \text{if } b_j > \hat{b}_{ij} 
\end{cases}
$$

where $\hat{b}_{ij}$ refers to the bid such that advertiser $i$’s budget is just enough to ensure $i$’s duration in the auction till the end of game period given that advertiser $j$’s budget is exhaustible. In other words, $t_1(\hat{b}_{ij}) + t_2(\hat{b}_{ij}) = 1$. We now explain why $\Pi_i(b_i,b_j|b_i < b_j)$ might take three functional forms. When $b_j$ is so small that advertiser $i$ has no jamming intention, advertiser $i$’s profit takes the lowest form $\pi_i - r$. When $b_j$ exceeds $\frac{B_i}{\sigma}$ but is moderate, advertiser $i$ will not deplete $B_j$ too soon by bidding $b_j - \varepsilon$. In other words, advertiser $i$ will not exit the auction during the day. However, when $b_j > \hat{b}_{ij}$, advertiser $i$ can only obtain a fraction of clicks that her budget can afford, which is equal to $\frac{B_i}{r}$. By definition, we have $\hat{b}_{ij} > \frac{B_i}{\sigma}$ provided that $B_i > r$.

Next we derive advertiser $i$’s UBRF. In case (a) where $B_j < \frac{\sigma(\sigma r - B_i)}{\sigma - 1}$, Lemma D2 indicates $r > \hat{b}_{ij}$. Since $b_j \geq r > \hat{b}_{ij}$, advertiser $i$’s profit function is the same as the case in which $B_i < r$, which implies $b_i(b_j) = r$.

In case (b), we first show that there must be $\hat{B}_i > \frac{\sigma(\sigma r - B_i)}{\sigma - 1}$. We have $\hat{B}_i = \frac{\sigma \pi_i b_i}{\pi_i + B_i - r} > \sigma r$ because $\hat{B}_i$ increases with $\pi_i$ and $\pi_i > r$. Since $B_i > r$, we also have $\frac{\sigma \pi_i B_i}{\pi_i + B_i - r} < \frac{\sigma(\sigma r - r)}{\sigma - 1} = \sigma r$, which proves that $\hat{B}_i > \frac{\sigma(\sigma r - B_i)}{\sigma - 1}$ as long as $B_i > r$. Next we show that any $b_i \in [r, \hat{b}_{ij})$ is weakly dominated by $\check{b}_{ij}$ and any $b_i \in (\hat{b}_{ij}, +\infty)$ is weakly dominated by $\hat{b}_{ij}$. For the former statement, we consider $b_i \in [r, \hat{b}_{ij})$, then $\Pi_i$ is the same for $b_i$ and $\hat{b}_{ij}$ when $b_i < \hat{b}_{ij}$. When $b_j > \hat{b}_{ij}$, since $B_j < \hat{B}_i$, Lemma C2 indicates $B_j < \hat{b}_{ij}$, which suggests $\Pi_i(\check{b}_{ij}) = (\pi_i - r) \frac{B_j}{\sigma b_i} + \sigma(\pi_i - r) \left(1 - \frac{B_j}{\sigma b_i} \right) > (\pi_i - r) \frac{B_j}{\sigma b_i} + \sigma(\pi_i - r) \left(1 - \frac{B_j}{\sigma b_i} \right) \geq \Pi_i(b_i)$. When $b_j$ lies in the middle of $b_i$ and $\check{b}_{ij}$, $\Pi_i(\check{b}_{ij}) = (\pi_i - b_j) \frac{B_i}{b_j} \geq (\pi_i - \hat{b}_{ij}) \frac{B_i}{\hat{b}_{ij}} = (\pi_i - r) \frac{B_j}{\sigma b_i} + \sigma(\pi_i - r) \left(1 - \frac{B_j}{\sigma b_i} \right) > (\pi_i - r) \frac{B_j}{\sigma b_i} + \sigma(\pi_i - r) \left(1 - \frac{B_j}{\sigma b_i} \right) \geq \Pi_i(b_i)$. Thus, we have shown that any $b_i \in [r, \hat{b}_{ij})$ is weakly dominated by $\check{b}_{ij}$. Similarly, one can also check that any $b_i \in (\check{b}_{ij}, +\infty)$ is weakly dominated by $\hat{b}_{ij}$.

Now we are ready to verify the UBRF in case (b). When $b_j < \check{b}_{ij}$, the profit of staying at the first position is $$(\pi_i - b_j) \frac{B_i}{b_j} \geq (\pi_i - \check{b}_{ij}) \frac{B_i}{\check{b}_{ij}} = (\pi_i - r) \frac{B_j}{\sigma b_i} + \sigma(\pi_i - r) \left(1 - \frac{B_j}{\sigma b_i} \right) > (\pi_i - r) \frac{B_j}{\sigma b_i} + \sigma(\pi_i - r) \left(1 - \frac{B_j}{\sigma b_i} \right),$$ which is the highest possible profit of staying at the second position. Thus,
advertiser \(i\) should choose to bid above \(b_j\) when \(b_j < \tilde{b}_{ij}\). Given that any bid below \(\tilde{b}_{ij}\) is weakly dominated by \(\tilde{b}_{ij}\), the UBRF in this case is \(b_i(b_j) = \frac{\sigma}{\sigma - 1} B_i\). When \(b_i \leq b_j \leq \tilde{b}_{ij}\), the profit at the second position equals \((\pi_i - r)\frac{B_j}{\sigma b_i} + \sigma(\pi_i - r)\left(1 - \frac{B_i}{\sigma b_i}\right)\), which exceeds the profit at the first and increases with \(b_i\). Thus, \(b_i(b_j) = b_j - \epsilon\). When \(b_j > \tilde{b}_{ij}\), the profit at the second position is

\[
\left\{ \begin{array}{ll}
(\pi_i - r)\frac{B_j}{\sigma b_i} + \sigma(\pi_i - r)\left(1 - \frac{B_i}{\sigma b_i}\right) & \text{if } b_j < \tilde{b}_{ij}, \\
(\pi_i - r)\frac{B_i}{r} & \text{if } b_i > \tilde{b}_{ij}\end{array} \right.
\]

\(b_i \geq \tilde{b}_{ij}\). Because any bid above \(\tilde{b}_{ij}\) is weakly dominated by \(\tilde{b}_{ij}\), the UBRF in this case is therefore \(b_i(b_j) = \tilde{b}_{ij}\).

Finally, for case (c) in which \(B_j \geq \tilde{B}_i\), Lemma C2 indicates advertiser \(j\) has no jamming apprehension under \(\tilde{b}_{ij}\). Hence, the bid threshold below which all bids are weakly dominated by changes from \(\tilde{b}_{ij}\) to \(\tilde{b}_i\). Furthermore, as shown in the proof of Lemma C3, any bid \(b_j \in (\tilde{b}_j, \frac{B_j}{\sigma})\) is also weakly dominated by \(\tilde{b}_i\). Following the similar procedures in the proof of case (b), we can verify the UBRF in case (c).

**Lemma D4.** Suppose \(\pi_i \geq \pi_j\), when \(B_i \in \left[\max\left\{B_j, \frac{\sigma(\pi - B_j)}{(\pi - 1)}\right\}, \tilde{B}_j\right]\) and \(B_j \in \left[\max\left\{r, \frac{\sigma r}{\sigma - 1}\right\}, \min\{B_i, \frac{\sigma r}{\sigma - 1}\}\right]\) (the left part of area 13 in Figure B1(b)), we have \(\tilde{b}_{ij} \geq \tilde{b}_j\).

**Proof of Lemma D4.** Following the logic used in the proof of Lemma C5, we only need to prove that given any \(B_j\) within its range, \(\tilde{b}_{ij}(B_i) \geq \tilde{b}_j(B_i)\) holds at the lower- and upper-boundaries of \(B_i\). We first look at the case in which \(B_j \geq \frac{\sigma^2 r}{\sigma - 1}\) (i.e., \(B_j \geq \frac{\sigma(\pi - B_j)}{(\pi - 1)}\)). Following exactly the same procedures in the proof of Lemma C5, we can show that \(\tilde{b}_{ij}(B_i) \geq \tilde{b}_j(B_i)\) when \(B_i = B_j\) or \(B_i = \tilde{B}_j\). As for the case in which \(B_j < \frac{\sigma^2 r}{\sigma - 1}\), the upper bound of \(B_i\) remains the same while the lower bound changes from \(B_j\) to \(\frac{\sigma(\pi - B_j)}{(\pi - 1)}\). Next we show \(\tilde{b}_{ij}(B_i) \geq \tilde{b}_j(B_i)\) when \(B_i = \frac{\sigma(\pi - B_j)}{(\pi - 1)}\). When \(B_i = \frac{\sigma(\pi - B_j)}{(\pi - 1)}\), \(\tilde{b}_j = r\) and it is easy to check that \(B_j \geq \frac{\sigma(\pi - B_j)}{(\pi - 1)}\) (because the curve \(B_i = \frac{\sigma(\pi - B_j)}{(\pi - 1)}\) lies above \(B_j = \frac{\sigma(\pi - B_j)}{(\pi - 1)}\) in this region), which indicates \(\tilde{b}_{ij} \geq r = \tilde{b}_j\). This completes our proof of Lemma D4.

**Proof of Proposition 2.** We first verify the UNE in budget areas 10–19 described in Table B3. In area 10 where \(B_i \leq \frac{\sigma(\pi - B_j)}{\sigma - 1}\) & \(B_j \leq \frac{\sigma(\pi - B_i)}{\sigma - 1}\), Lemma D3 indicates that both advertisers bid \(r\) in equilibrium. Area 11 where \(B_i \in \left[\sigma r - \frac{\sigma}{\sigma - 1} B_j, \tilde{B}_j\right]\) & \(B_j \in \left(0, \max\left\{r, \sigma r - \frac{\sigma}{\sigma - 1} B_i\right\}\right]\) actually consist of two subcases. In the left part where \(B_i \leq \sigma r\), since \(B_j \leq \sigma r - \frac{\sigma}{\sigma - 1} B_i\) is equivalent to \(B_i \leq \frac{\sigma(\pi - B_j)}{(\pi - 1)}\), it implies \(b_j = r\) based on Lemma D1 and D3. Applying Lemma D3 to advertiser \(i\) further indicates that any \(b_i \in \left[\tilde{b}_{ij}, \tilde{b}_{ij}\right]\) and \(b_j = r\) is the UNE in this left subcase. As for the right part where \(B_i \in (\sigma r, \tilde{B}_j]\) & \(B_j \in (0, r]\), we know \(b_j = r\) from Lemma D1. On the other hand, since \(B_i\) is not too large, advertiser \(i\) has both jamming apprehension and intention, which implies that any bid above \(\tilde{b}_{ij}\) is weakly
undominated for advertiser $i$. Hence, any $b_i \in [\bar{b}_{ij}, +\infty)$ and $b_j = r$ is the UNE in the right subcase. In area 12 where $B_i \in (\bar{B}_{ij}, +\infty)$ & $B_j \in (0, r]$, $b_j = r$ from Lemma D1 and $b_i \in [\bar{b}_{ij}, +\infty)$ in UNE because advertiser $i$ has no jamming apprehension but intention. In area 14 where $B_i \in (\bar{B}_{ij}, +\infty)$ & $B_j \in (r, \sigma r]$, similar to the case in area 12, advertiser $i$ will not bid below $\bar{b}_{ij}$. Also, from Lemma D3, we can see that any $b_i > \frac{B_i}{\sigma}$ cannot sustain in equilibrium. The reason is that any $b_i > \frac{B_i}{\sigma}$ will trigger the jamming war and drag the highest bid down to the level no larger than $\frac{B_i}{\sigma}$. Given $b_i \in [\bar{b}_{ij}, \frac{B_i}{\sigma}]$ in the UNE, Lemma D2 indicates that $b_j = \hat{b}_j$ because $B_j \in (r, \sigma r]$ and $B_i \geq \bar{b}_{ij} > \hat{b}_j$.

Next we derive the UNE in a more complicated case, area 13, which can be further separated into three parts: the left part where $B_i < \sigma r$, the middle part where $B_i \in [\sigma r, \bar{B}_{ij}]$ and the right part where $B_i \in [\bar{B}_{ij}, \bar{B}_j]$. We first prove that the advertiser with a higher $\bar{b}_{ij}$ ($\bar{b}_{ji}$) wins the first position in any UNE. Lemma D3 says $\bar{b}_{ij} \geq \bar{b}_{ji}$ in the left and middle part of area 13 and Lemma C2 says $\bar{b}_{ij} \geq \bar{b}_{ji}$ because $B_i \in [\bar{B}_{ij}, \bar{B}_j]$. Thus, we have $\bar{b}_{ij} \geq \bar{b}_{ji}$ in area 13. Suppose advertiser $j$ bids higher at equilibrium, then $b_j > b_i \geq \bar{b}_{ij}$ because $\bar{b}_{ij}$ is the lowest weakly undominated bid for advertiser $i$. Since $\bar{b}_{ij} \geq \bar{b}_{ji}$, Lemma D3 tells us that advertiser $j$’s best response is bid below $b_i$, which contradicts with $b_j > b_i$. Thus, we have proved advertiser $i$ bids higher in any UNE here.

In the middle and the right parts of area 13, the UBRF of advertiser $i$ is

$$b_i(b_j) = \begin{cases} x | x > \bar{b}_{ij} & \text{if } b_i < \bar{b}_{ij} \\ b_j - \epsilon & \text{if } \bar{b}_{ij} \leq b_j \end{cases}.$$}

Combined with the UBRF of advertiser $j$ described in Lemma D3, we can see that the UNE must be $(\bar{b}_{ij}, \min(\bar{b}_{ij} - \epsilon, \bar{b}_{ji}))$. Actually, we can prove that given any $B_i \in [\sigma r, \bar{B}_{ij}]$, there must exist a threshold of $B_j$ denoted by $B_j^\#$ such that $b_j = \hat{b}_{ji}$ when $B_j \in (r, B_j^\#)$ and $b_j = \bar{b}_{ij} - \epsilon$ when $B_j \in [B_j^\#, \sigma r]$ in the UNE. Too see this, first notice that given $B_i$, $\hat{b}_{ji} > \bar{b}_{ij}$ is equivalent to $f(B_j) = (\sigma - 1)(\pi_i - r)B_j^2 + [\pi_iB_i - (\sigma - 1)\pi_i(\pi_i - r)]B_j + (\sigma - 1)\pi_iB_i^2 - [(\sigma - 1)r + \pi_i]\sigma B_i > 0$. The FOC is $f'(B_j) = 2(\sigma - 1)(\pi_i - r)B_j + \sigma \pi_iB_i - (\sigma - 1)(\pi_i - r)\sigma$, the right-hand side of which increases with $\pi_i$ because $B_i \geq \sigma r$. Hence, we have $f'(B_j) \geq 2(\sigma - 1)(r - r)B_j + \sigma \pi_iB_i - (\sigma - 1)(r - r)\sigma \geq \sigma \pi_iB_i > 0$. Next we prove $\hat{b}_{ji} > \bar{b}_{ij}$ when $B_j = \sigma r$ and $\hat{b}_{ji} \leq \bar{b}_{ij}$ when $B_j = r$. When $B_j = \sigma r$, $\hat{b}_{ji} = +\infty > \bar{b}_{ij}$. When $B_j = r$, $\hat{b}_{ji} = \frac{\bar{B}_i}{\sigma} \leq \bar{b}_{ij}$ because $B_i \leq \bar{b}_{ij}$ and the result from Lemma C2. Thus, we have shown that $f(r) \leq 0$, $f(\sigma r) > 0$ and $f' > 0$. Thus, there must exists a $B_j^\# \in [r, \sigma r]$ s.t. $f(B_j^\#) = 0$.

The similar analysis can be applied to the left part of area 13 where $B_i < \sigma r$ & $B_j \in [\sigma r - \frac{(\sigma - 1)B_i}{\sigma} B_i]$. The only twist is in the proof of $f'(B_j) > 0$. Since $B_j \geq \sigma r - \frac{(\sigma - 1)B_i}{\sigma} > \frac{\sigma(B_i - B_i)}{\sigma - 1}$, we have $f'(B_j) = 2(\sigma - 1)(\pi_i - r)B_j + \sigma \pi_iB_i - (\sigma - 1)(\pi_i - r)\sigma > 2\sigma(\pi_i - r)(\sigma - B_i) + \sigma \pi_iB_i - (\sigma - 1)(\pi_i - r)\sigma$, the right-hand side of which increases with $\pi_i$ because $2\sigma(\sigma - B_i) + \sigma B_i - (\sigma - 1)\sigma r = \sigma(\pi_i - B_i) + \sigma r > 0$. Thus, we have shown $f'(B_j) > 0$. At the new boundaries, we show that given $B_i$, $f(B_j) > 0$ when $B_j = B_i$ and $f(B_j) < 0$ when $B_j = \sigma r - \frac{(\sigma - 1)B_i}{\sigma}$. When $B_j = B_i$, $b_j = \hat{b}_{ij} > \bar{b}_{ij}$ because $\pi_i > \bar{b}_{ij} \iff B_j > \frac{\sigma(B_i - B_i)}{\sigma - 1}$ and $B_i \geq \sigma r - \frac{(\sigma - 1)B_i}{\sigma}$ implies $B_j > \frac{\sigma(B_i - B_i)}{\sigma - 1}$. When $B_j = \sigma r - \frac{(\sigma - 1)B_i}{\sigma}$, $b_j = r < \bar{b}_{ij}$ because Lemma D2 says $r < \bar{b}_{ij} \iff B_j > \frac{\sigma(B_i - B_i)}{\sigma - 1}$ and $\sigma r - \frac{(\sigma - 1)B_i}{\sigma}$ in the left part of area 13.
So far, we have completed the UNE derivation in the half budget space where \( B_i \geq B_j \). The UNE in the other half space (i.e., areas 15 to 19) can be obtained in a similar way.

The analysis above indicates the following expressions for budget regions of different bid strategies: \( \Theta_{A_i} = \{\text{Area 13 & 15 where } \tilde{b}_{ij} \in [\bar{b}_{ij}, \tilde{b}_{ij}]\} \), \( \Theta_{A_j} = \{\text{Area 16 & 15 where } \tilde{b}_{ij} \in (\tilde{b}_{ij}, \bar{b}_{ij}]\} \), \( \Theta_{D_i} = \{\text{Area 17}\} \), \( \Theta_{D_j} = \{\text{Area 14}\} \), \( \Theta_R = \{\text{Area 10,11,12,18 & 19}\} \), \( \Theta_{P_i} = \{\text{Area 13 & 15 where } \tilde{b}_{ij} \geq \bar{b}_{ij} & \tilde{b}_{ij} > \bar{b}_{ij}\} \) and \( \Theta_{P_j} = \{\text{Area 16 & 15 where } \tilde{b}_{ij} > \bar{b}_{ij} & \tilde{b}_{ij} > \bar{b}_{ij}\} \). Statements regarding the conditions for advertisers’ uses of various bid strategies are direct from the UNE described in Table B3.

**Corollary D1.** Both the upper and the lower bounds of advertisers’ bids weakly increase with their budgets.

**Proof of Corollary D1.** We first prove that the lower bound of \( b_i \) weakly increases with \( B_i \) in two steps. First, note that all possible types of equilibrium bids for advertiser \( i (b_i^*, \tilde{b}_i, \tilde{b}_{ij} - \epsilon, \tilde{b}_{ij}, \tilde{b}_{ij}) \) weakly increase with \( B_i \) within the corresponding budget area. The positive relationships between \( \tilde{b}_{ij} \), \( \tilde{b}_{ij} \) and \( B_i \) are obvious. \( \tilde{b}_{ij} \) is independent of \( B_i \) by definition. As for \( \tilde{b}_i \) and \( \tilde{b}_{ij} \), we find that \( \frac{\partial \tilde{b}_i}{\partial B_i} \geq 0 \) is equivalent to \( B_i \leq B_i^* \) and \( \frac{\partial \tilde{b}_{ij}}{\partial B_i} \geq 0 \) is equivalent to \( B_j \leq B_i^* \). These conditions are satisfied in areas where \( b_i = \tilde{b}_i \) or \( \tilde{b}_{ij} \) at equilibrium. The second step is to check whether the lower bound of \( b_i \) weakly increases when \( B_i \) lies on the boundary of two adjacent budget areas. For example, on the boundary of area 10 and 11, \( b_i = r = \tilde{b}_{ij} \) because \( B_j = \frac{\sigma(\sigma - B_i)}{\sigma - 1} \). It can be verified that the lower bound of \( b_i \) is either unchanged or strictly increases at the boundary of other adjacent areas. Thus, we have proved that the lower bound of \( b_i \) weakly increases with \( B_i \) in the whole budget space. The positive relationship between the upper bound of \( b_i \) and \( B_i \) can also be verified following the same procedure. Finally, a similar analysis can be conducted for advertiser \( j \) to check the positive relationship between her bid and budget.

**Appendix E. Revenue Analysis**

**Proof of Proposition 3.** We calculate each advertiser’s profit in each budget area and summarize the results in Table E1. We describe the relationship between advertisers’ profits and budgets in two steps. For each advertiser, saying advertiser \( i \), we first show that within each budget area, \( \Pi_i \) weakly increases with \( B_i \) except for the areas 13 and 15 where \( \tilde{b}_{ij} < \bar{b}_{ij} \). Table E1 shows that \( \Pi_i \) is independent of \( B_i \) in areas 3, 5, 6, 7, 8, 12, 14, 17. \( \Pi_i \) takes four different forms in the rest of budget areas. In area 2 and part of 1, 15, 16, \( \Pi_i = (\pi_i - r) \frac{B_j}{\sigma b_{ij}} + \sigma(\pi_i - r)(1 - \frac{B_j}{\sigma b_{ij}}) \), which increases with \( \tilde{b}_{ij} \). Since \( \tilde{b}_{ij} \) increases with \( B_i \), we have proved \( \Pi_i \) increases with \( B_i \) in these areas. In area 4, 9 and part of 1, 13, 15, \( \Pi_i = \frac{\pi_i B_i}{\tilde{b}_{ij}} - B_i = \frac{(\pi_i - r)\sigma \pi_i - (1-\frac{1}{\sigma})B_j}{\pi_i(\pi_i - r)(1-\frac{1}{\sigma})b_i} \), which increases with \( B_i \). In area 10, 18, 19 and part of 11, 15, 16, \( \Pi_i = \frac{\pi_i B_i}{r} - B_i \), which increases with \( B_i \) because \( \pi_i > r \). In area 13 and 15 where \( \tilde{b}_{ij} < \bar{b}_{ij} \), \( \Pi_i = \frac{\pi_i B_i}{\tilde{b}_{ij}} - \frac{\sigma \pi_i (\sigma - B_i)}{(\sigma - 1)r} - B_i \), which decreases with \( B_i \). The second step is to prove \( \Pi_i \) weakly increases at the
boundary of two adjacent areas. Actually, it is easy to verify that $\Pi_i$ remains the same on all boundaries except the boundaries of 5 & 6, 9 & 7, 9 & 8, 13 & 14, at which $\Pi_i$ strictly increases with $B_i$ due to the decrease in CPC. So far, we have finished the proof regarding the relationship between $\Pi_i$ and $B_i$. The proof for the relationship between $\Pi_j$ and $B_j$ is similar and therefore omitted. $\square$

Table E1. Advertisers’ Profits and Publisher’s Revenue

<table>
<thead>
<tr>
<th>Area No. in Fig. B1</th>
<th>$\Pi_i$</th>
<th>$\Pi_j$</th>
<th>$\Pi_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 &amp; 1 where $\bar{b}<em>{ij} &gt; \bar{b}</em>{ij}$</td>
<td>$(\pi_i - r) \frac{B_j}{\sigma b_{ij}} + \alpha(\pi_i - r)(1 - \frac{B_j}{\sigma b_{ij}})$</td>
<td>$\frac{\pi_j B_{ji}}{b_{ji}} - B_j$</td>
<td>$B_j + \left(1 - \frac{B_j}{\sigma b_{ij}}\right)\sigma r + \frac{B_{ji}}{\sigma b_{ij}}$</td>
</tr>
<tr>
<td>3 &amp; 17</td>
<td>$\pi_i - r$</td>
<td>$\sigma (\pi_i - \bar{b}_{ij})$</td>
<td>$\bar{b}_{ij} + r$</td>
</tr>
<tr>
<td>4, 9 &amp; 1 where $\bar{b}<em>{ij} \leq \bar{b}</em>{ij}$</td>
<td>$(\pi_j - r) \frac{B_{ij}}{\sigma b_{ij}} + \alpha(\pi_j - r)(1 - \frac{B_{ij}}{\sigma b_{ij}})$</td>
<td>$B_i + \left(1 - \frac{B_{ij}}{\sigma b_{ij}}\right)\sigma r + \frac{B_{ij}}{\sigma b_{ij}}$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$\pi_i - r$</td>
<td>$(\pi_j - r) \frac{B_{ij}}{\sigma b_{ij}} + \alpha(\pi_j - r)(1 - \frac{B_{ij}}{\sigma b_{ij}})$</td>
<td>$B_i + \left(1 - \frac{B_{ij}}{\sigma b_{ij}}\right)\sigma r + \frac{B_{ij}}{\sigma b_{ij}}$</td>
</tr>
<tr>
<td>6 &amp; 7</td>
<td>$\sigma (\pi_i - b_{ij})$</td>
<td>$\pi_j - r$</td>
<td>$B_j + r$</td>
</tr>
<tr>
<td>8 &amp; 14</td>
<td>$\sigma (\pi_i - \bar{b}_{ij})$</td>
<td>$\pi_j - r$</td>
<td>$B_j + r$</td>
</tr>
<tr>
<td>10</td>
<td>$\pi_i B_{ij} - B_i$</td>
<td>$\pi_j - r$</td>
<td>$B_i + B_j$</td>
</tr>
<tr>
<td>11</td>
<td>$\frac{\pi_i B_{ij}}{r} - B_i \text{ if } B_i &lt; \sigma r$</td>
<td>$\frac{\pi_j B_{ij}}{r} - B_j \text{ if } B_j &lt; \sigma r$</td>
<td>$B_i + B_j \text{ if } B_i &lt; \sigma r$</td>
</tr>
<tr>
<td>12</td>
<td>$\sigma (\pi_i - r)$</td>
<td>$\sigma (\pi_j - r)$</td>
<td>$B_i + \sigma r \text{ if } B_i \geq \sigma r$</td>
</tr>
<tr>
<td>13 &amp; 15 where $\bar{b}<em>{ij} \leq \bar{b}</em>{ij}$</td>
<td>$(\pi_i - r) \frac{B_{ij}}{\sigma b_{ij}} + \alpha(\pi_i - r)(1 - \frac{B_{ij}}{\sigma b_{ij}})$</td>
<td>$\frac{\pi_j B_{ij}}{B_{ij}} - B_j \text{ if } \bar{b}<em>{ij} \geq \tilde{b}</em>{ij}$</td>
<td>$B_i + \left(1 - \frac{B_{ij}}{\sigma b_{ij}}\right)\sigma r + \frac{B_{ij}}{\sigma b_{ij}} \text{ if } \bar{b}<em>{ij} \geq \tilde{b}</em>{ij}$</td>
</tr>
<tr>
<td>16 &amp; 15 where $\bar{b}<em>{ij} &gt; \bar{b}</em>{ij}$</td>
<td>$(\pi_i - r) \frac{B_{ij}}{\sigma b_{ij}} + \alpha(\pi_i - r)(1 - \frac{B_{ij}}{\sigma b_{ij}})$</td>
<td>$\frac{\pi_j B_{ij}}{B_{ij}} - B_j \text{ if } \bar{b}<em>{ij} \geq \tilde{b}</em>{ij}$</td>
<td>$B_i + \left(1 - \frac{B_{ij}}{\sigma b_{ij}}\right)\sigma r + \frac{B_{ij}}{\sigma b_{ij}} \text{ if } \bar{b}<em>{ij} \geq \tilde{b}</em>{ij}$</td>
</tr>
<tr>
<td>18</td>
<td>$\frac{\pi_j B_{ij}}{r} - B_i \text{ if } B_i &lt; \sigma r$</td>
<td>$\sigma (\pi_j - r)$</td>
<td>$B_i + B_j \text{ if } B_i &lt; \sigma r$</td>
</tr>
<tr>
<td>19</td>
<td>$\frac{\pi_j B_{ij}}{r} - B_i \text{ if } B_i \geq \sigma r$</td>
<td>$\sigma (\pi_j - r)$</td>
<td>$B_i + \sigma r \text{ if } B_i \geq \sigma r$</td>
</tr>
</tbody>
</table>

**Proof of Proposition 4.** We report the publisher’s revenue $\Pi_p$ in each budget area in Table E1. We prove the statements made in Proposition 4 in several steps. First, we show that $\Pi_p$ weakly increases with $B_i$ and $B_j$ in each budget area except area 1 and 15. We only prove several less obvious cases. For
example, in area 2, \( \Pi_p = B_j + \left(1 - \frac{B_j}{\sigma b_j} \right) \sigma r + \frac{B_j r}{\sigma b_j} = \sigma r + B_j \left(1 - \frac{(\sigma - 1)r}{\sigma b_j} \right) \). Because \( \tilde{b}_{ji} \) increases with both \( B_i \) and \( B_j \), it is easy to see that \( \Pi_p \) increases with both \( B_i \) and \( B_j \) here. In area 5, \( \Pi_p = \sigma r + B_i \left(1 - \frac{(\sigma - 1)r}{\sigma b_i} \right) \), which increases with \( B_i \) because \( \tilde{b}_{ij} \) increases with \( B_i \) in this area.

Second, based on the relationship between CPC and budgets, we can easily verify that given any \( B_j > r \), \( \Pi_p \) strictly decreases when \( B_i \) exceeds \( \min \{ \tilde{b}_{ij}, B_i^* \} \) and given any \( B_i \in (r, \tilde{b}_{ij}) \), \( \Pi_p \) strictly decreases when \( B_j \) exceeds \( \tilde{b}_{ji} \).

Finally, we prove that \( \Pi_p \) strictly decreases with \( B_i \) when \( B_i \) reaches the threshold under which position paradox disappears. Position paradox disappears at three boundaries: 1) when \( B_i \) increases from area 3 to 5; 2) when \( B_i \) increases so that \( \tilde{b}_{ij} = \tilde{b}_{ij} \) in area 1; 3) when \( B_i \) increases so that \( \tilde{b}_{ij} = \tilde{b}_{ij} \) in area 15. In the first case, \( \Pi_p \) (area 5) − \( \Pi_p \) (area 3) = \( B_j + \left(1 - \frac{B_j}{\sigma b_j} \right) \sigma r + \frac{B_j r}{\sigma b_j} - \tilde{b}_{ij} - r = (\tilde{b}_{ij} - B_j) \left(\frac{(\sigma - 1)r}{B_i} - 1\right) < 0 \) because \( \tilde{b}_i - B_i > 0 \) and \( \tilde{b}_j > B_i > \sigma r > (\sigma - 1)r \). In the second case when \( \tilde{b}_{ji} = \tilde{b}_{ij} \) in area 1, \( \Pi_p \) changes from \( B_j + \left(1 - \frac{B_j}{\sigma b_j} \right) \sigma r + \frac{B_j r}{\sigma b_j} \) to \( B_i + \left(1 - \frac{B_i}{\sigma b_i} \right) \sigma r + \frac{B_i r}{\sigma b_i} \). The revenue difference is \( B_j \left(\frac{1}{\sigma b_j} - \frac{(\sigma - 1)r}{\sigma b_j} \right) - B_j \left(\frac{1}{\sigma b_j} - \frac{(\sigma - 1)r}{\sigma b_j} \right) = (B_i - B_j) \left(1 - \frac{(\sigma - 1)r}{\sigma b_i} \right) < 0 \) because \( B_i < B_j \) in area 1 where \( \tilde{b}_{ji} = \tilde{b}_{ij} \). In the third case when \( \tilde{b}_{ji} = \tilde{b}_{ij} \) in area 15, Lemma D2 indicates \( b_j \geq \tilde{b}_{ij} = \tilde{b}_{ij} \) and \( \tilde{b}_{ji} \geq \tilde{b}_{ji} = \tilde{b}_{ij} \). Hence, \( \Pi_p \) changes from \( B_j + \left(1 - \frac{B_j}{\sigma b_j} \right) \sigma r + \frac{B_j r}{\sigma b_j} \) to \( B_i + \left(1 - \frac{B_i}{\sigma b_i} \right) \sigma r + \frac{B_i r}{\sigma b_i} \). Similar to the case in area 1, \( \Pi_p \) falls because advertiser \( i \) with a smaller budget stays at the first position when \( B_i \) exceeds the boundary of \( \tilde{b}_{ji} = \tilde{b}_{ij} \). This concludes our proof of Proposition 4. □

**Appendix F. Equilibrium Derivation and Proofs for Endogenous Advertising Budgets**

By solving the game backwards, we use \( \Pi_k^{mn} \) to denote advertiser \( k \)'s total advertising profit when \( i \) chooses \( B^m \) and \( j \) chooses \( B^n \), where \( k = i, j \) and \( m, n = H, L \). All \( \Pi_k^{mn} \) here can be calculated from the UNE described in Table B3.

**Outline of the proof of Proposition 5.** We focus on the case in which \( \alpha \) is relatively small. We first characterize how advertisers’ budget choices vary with \( c \) when the difference between their value-per-clicks is small (Lemma F1) or large (Lemma F2). After that, we prove that the publisher’s revenue has an inverted-U relationship with \( c \) (Lemma F3).

**Lemma F1.** When \( \frac{\pi_i}{\pi_j} \in (1, \delta] \) and \( \alpha \leq \min \left\{ \frac{b_{ij}}{B_i^*}, \alpha_1, \alpha_2 \right\} \), the budget choices of advertisers at equilibrium are described below.

1. When \( c < c_1 \), \( B_i = B_i^H \) and \( B_j = B_j^L \);
2. When \( c_1 \leq c \leq c_2 \), \( B_i = B_i^L \) and \( B_j = B_j^H \);
3. When \( c > c_2 \), \( B_i = B_i^L \) and \( B_j = B_j^L \),

\[
\frac{(\sigma + \alpha - \alpha) \pi_j}{\alpha \pi_j} \left( \frac{1}{1 - \alpha} \right), \quad c_2 = \frac{(\sigma + \alpha - \alpha) \pi_i}{\alpha \pi_i} \left( \frac{1}{1 - \alpha} \right), \quad c_3 = \left( \frac{\pi_i - \pi_j}{\pi_i} \right) \delta, \quad c_3 = \left( \frac{c_3}{1 - \alpha} \right)
\]

where \( c_1 = \left( \frac{\sigma \pi_j + (\sigma - 1) \pi_j}{1 - \alpha} \right), \ c_2 = \left( \frac{\sigma \pi_j + (\sigma - 1) \pi_j}{1 - \alpha} \right), \ c_3 = \left( \frac{\pi_i - \pi_j}{\pi_i} \right) \delta \) is the threshold s.t. \( c_1 = c_3 \) as functions of \( \frac{\pi_i}{\pi_j} \), \( \alpha_1 \) is the threshold s.t. \( c_1 = c_2 \) as functions of \( \alpha \) and \( \alpha_2 \) is the threshold s.t. \( c_2 = c_3 \) as functions of \( \alpha \).
Proof of Lemma F1. When both advertisers choose high budgets, advertiser $i$ stays at the first position and pays $b_j^*$ for each click. Therefore, we have $\Pi_{i}^{HH} = \sigma \left( \pi_i - b_j^* \right) - cB_j^*$ and $\Pi_{j}^{HH} = \pi_j - cB_j^*$. Similarly, we can derive that $\Pi_{i}^{HL} = \sigma \left( \pi_i - \bar{b}_i (\alpha B_j^*) \right) - cB_j^*$, $\Pi_{j}^{HL} = \pi_j - c\alpha B_j^*$. When advertiser $i$ chooses low budget and $j$ chooses high budget, the condition $\alpha < \frac{B_{ij}^l}{\bar{b}_i^l}$ indicates the UNE is in area 3. Thus, we have $\Pi_{i}^{IH} = \pi_i - c\alpha B_j^*$, $\Pi_{j}^{IH} = \sigma \left( \pi_j - \bar{b}_i (\alpha B_j^*) \right) - cB_j^*$. When both choose low budgets, bid jamming occurs and we have $\Pi_{i}^{LL} = \sigma \left( \pi_i - \bar{b}_i (\alpha B_j^*) \right) - c\alpha B_j^*$, and $\Pi_{j}^{LL} = \pi_j \frac{\alpha B_j^l}{\bar{b}_j^l} + \sigma \pi_j \left( 1 - \frac{\alpha B_j^l}{\bar{b}_j^l} \right) - c\alpha B_j^*$, where $\bar{b}_i (\alpha B_j^*) = \frac{\pi_i (\alpha B_j^* + (1 - \frac{\alpha}{\pi_i}) \pi_j \alpha B_j^*)}{\pi_i + \sigma \pi_i}$ and $\bar{b}_i (\alpha B_j^*) = \frac{\pi_i \alpha B_j^l}{\pi_i + \alpha B_j^l}$.

By definition, we have $\Pi_{i}^{HH} < \Pi_{i}^{HL}, \Pi_{i}^{HL} > \Pi_{i}^{LL}$ is equivalent to $c < c_1$, $\Pi_{i}^{HL} > \Pi_{i}^{LH}$ is equivalent to $c < c_2$, and $\Pi_{i}^{HH} > \Pi_{i}^{HL}$ is equivalent to $c < c_3$. Next we prove that when $\alpha$ is relatively small, there must be $c_1 < c < c_2$. To see this, considering the extreme case in which $\alpha = 0$, then $\pi_i = \frac{\sigma \pi_j}{\sigma \pi_i + (\sigma - 1) \pi_j} < \frac{\pi_i}{\pi_i + (\sigma - 1) \pi_j} = c_2$, and $c_3 = \frac{\pi_i - \pi_j}{\pi_i} = 1 - \frac{\pi_j}{\pi_i} = 1 - \frac{(\sigma - 1) \pi_j}{(\sigma - 1) \pi_i} < 1 - \frac{(\sigma - 1) \pi_j}{\sigma \pi_i + (\sigma - 1) \pi_j} = c_2$. Thus, as long as $\alpha$ is not too large, we have $c_1 < c < c_2$. However, the relationship between $c_1$ and $c_3$ is still uncertain. It is easy to check that there exists a threshold $\delta$ such that $c_1 \geq c_3$ if and only if $\frac{\pi_i}{\pi_j} \leq \delta$.

Now we are ready to verify the equilibrium. When $c < c_1$, we know $\Pi_{i}^{HL} > \Pi_{i}^{LL}$ and $\Pi_{i}^{HH} < \Pi_{j}^{HL}$, which indicates that $(B_i^H, B_j^I)$ is the only equilibrium. When $c_1 \leq c < c_2$, we have $\Pi_{i}^{HH} \leq \Pi_{i}^{LL} \Pi_{j}^{HL} > \Pi_{j}^{LL}$, which suggest $(B_i^L, B_j^H)$ in equilibrium. The equilibrium when $c \geq c_2$ is both advertisers selecting the low budget. □

Lemma F2. When $\frac{\pi_i}{\pi_j} > \delta$ and $\alpha \leq \min \left\{ \frac{B_{ij}}{\bar{b}_i^l}, \alpha_1, \alpha_2 \right\}$, the budget choices of advertisers at equilibrium are described below.
1. When $c < c_1$, $B_i = B_i^H$ and $B_j = B_j^I$;
2. When $c_1 \leq c < c_2$, only mixed-strategy equilibrium exists and $\frac{\partial p_i^H}{\partial c} < 0$ while $\frac{\partial p_j^H}{\partial c} > 0$;
3. When $c_3 \leq c < c_2$, $B_i = B_i^I$ and $B_j = B_j^H$;
4. When $c \geq c_2$, $B_i = B_i^I$ and $B_j = B_j^L$,
where $c_1, c_2, c_3, \delta, \alpha_1, \alpha_2$ are defined in Lemma F1, and $p_i^H (p_j^H)$ denotes advertiser $i$’s (j’s) probability of choosing $B_i^H (B_j^H)$.

Proof of Lemma F2. The equilibrium derivation is similar to that in Lemma F1. The only difference is the case in which $c_1 \leq c < c_3$. In this case, it is easy to check that no pure-strategy equilibrium exists and therefore only mixed-strategy equilibrium exists. The mixed-strategy equilibrium $(p_i^H, p_j^H)$ is determined by the following conditions: both advertisers are indifferent between high and low budgets given the rival’s strategy. Thus, $p_i^H$ is determined by $p_i^H \Pi_{j}^{HH} + (1 - p_i^H) \Pi_{j}^{HL} = p_i^H \Pi_{j}^{LL} + (1 - p_i^H) \Pi_{j}^{HL}$, which implies that $p_i^H = \frac{\Pi_{j}^{HH} - \Pi_{j}^{HL}}{\Pi_{j}^{HH} - \Pi_{j}^{HL} + \Pi_{j}^{LL} - \Pi_{j}^{HH}}$. Since the numerator of $p_i^H$ decreases with $c$ and the denominator is a positive constant, we have proved that $\frac{\partial p_i^H}{\partial c} < 0$. Similarly, we can derive that $p_j^H =$
Because the numerator of $p_j^H$ increases with $c$ and the denominator is a positive constant, we have proved that $\frac{\partial p_j^H}{\partial c} > 0$. □

**Lemma F3.** The publisher’s revenue has an inverted-U relationship with the marginal budget cost.

**Proof of Lemma F3.** When $\frac{\pi_l}{\pi_j} \in (1, \delta]$, the publisher’s revenue first increases from $B_l^L$ to $\tilde{B}_l(B_l^L)$ when $c$ exceeds $c_1$, and then drops to $B_l^L$ when $c$ exceeds $c_2$. Thus, the publisher’s revenue first increases and then declines with $c$.

When $\frac{\pi_l}{\pi_j} > \delta$, we merely need to show that the publisher’s revenue is concave in $c$ when only mixed-strategy equilibrium exists. When $c_1 \leq c < c_3$, the publisher’s expected revenue is $E(\Pi) = p_l^H p_j^H B_l^* + p_l^L p_j^H (\alpha B_l^*) + p_l^L p_j^L (\alpha B_l^*) + p_l^L p_j^H (B_l^* - \alpha B_l^*) - p_l^H \alpha (B_l^* + B_l^*) - 2p_l^H \alpha B_l^*$. By definition, we have $\frac{\partial^2 p_l^H \partial p_j^H}{\partial c^2} = \frac{\partial^2 p_l^H}{\partial c^2} p_j^H + \frac{\partial^2 p_l^H}{\partial c^2} p_j^H + 2 \frac{\partial^2 p_l^H}{\partial c \partial c} p_j^H p_l^H$. Lemma F2 indicates that $2 \frac{\partial^2 p_l^H}{\partial c \partial c} p_j^H p_l^H < 0$. Furthermore, based on the functional form of $p_l^H$ and $p_j^H$, we know that they are both linear functions of $c$, which implies that $\frac{\partial^2 p_l^H}{\partial c^2} = \frac{\partial^2 p_l^H}{\partial c^2} = 0$. Thus, we have proved that $\frac{\partial^2 E(\Pi)}{\partial c^2} < 0$ when $c_1 \leq c < c_3$. The rest of the proof is similar to that in the case when $\frac{\pi_l}{\pi_j} > \delta$. □

**Proof of Proposition 5.** Lemmas F1 to F3 naturally lead to the conclusions made in Proposition 5. □

For the next two model extensions with incomplete information and stochastic outcome, we are unable to completely solve the bidding equilibrium. Instead, we check the robustness of our findings by describing the bidding equilibrium in budget areas where at least one advertiser has a sufficient budget.

**Appendix G: Model extensions**

We develop two model extensions to relax the assumptions in section 3. In the first extension we relax the complete information assumption and we explore position auctions with stochastic outcomes in the second extension. In all two cases, our objective is to show that the publisher’s revenue may decrease in an advertiser’s budget.

**G1. Position Auctions with Incomplete Information**

In section 3 we assume a complete information model. However, in reality it is possible that some start-up firms are newcomers to the market thus only limited knowledge about them is known. In order to further check the robustness of our findings, we study an incomplete information setup in which advertisers have uncertainty in regards to competitors’ budgets.

This extension is technically challenging. It has been shown in Benoît and Krishna (2001) that “one should not expect that the symmetric bidding strategy $b(B)$ in the incomplete information setting will be monotonically increasing and (therefore) standard differential equation techniques used to determine equilibrium strategies do not work.” In order to develop a tractable model with incomplete information, we make some reasonable simplifications in this extension.

First, we assume that each advertiser’s budget type is discrete: it can take only two points $B \in \{B^L, B^H\}$, where the high-budget option is above the sufficient budget for the high-value advertiser and the low-budget option is below the sufficient budget for the low-value advertiser. This assumption ensures that each advertiser has a positive probability to be budget constrained. Second, we assume that...
the low-budget option is not too small insofar as the position paradox does not occur at equilibrium. This allows us to focus on the budget area where the potential negative effect of budgets is driven by the decrease in CPC. Third, we assume that the probability of having a large budget is the same for both advertisers. Mathematically, it means \( \rho_l = \rho_j = \rho \), where \( \rho \) denotes the probability for advertiser \( i \) to take a high budget \( B^H \). Finally, in order to strengthen the robustness of this extension, we focus on a scenario in which advertisers have the least incentives to raise bids at the second position by assuming the highest external competition pressure, such that reservation value \( r \) is always one cent less than the second advertiser. Under these assumptions, we are able to find a Bayes-Nash equilibrium for this incomplete information game and summarize our main findings in the following proposition.

**Proposition 6.** When advertisers’ budgets are private knowledge, the publisher’s revenue strictly decreases when the expected budget of advertisers exceeds the threshold \( \rho^* B^H + (1 - \rho^*) B^L \), where

\[
\rho^* = \frac{\sigma \pi_j B^L}{(\sigma - 1) \pi_j B^L - B^L}
\]

This result further is consistent with our findings in the main model. When the expected budget of the high-value advertiser is small, the low-value advertiser prefers to stay at the second position with an optimal bid that is large enough to deplete \( B^L \) regardless of her budget type. However, when the expected budget of the high-value advertiser exceeds a threshold, the low-value advertiser will decrease her bid to the minimal level due to the external competition. This decrease in the low-value advertiser’s bid hurts the publisher’s revenue.

**G2. Position Auctions with Stochastic Outcomes**

Our main model focuses on a deterministic auction mechanism under which the position outcome is completely determined by the order of advertisers’ bids. This deterministic model is consistent with previous literature (e.g., Edelman et al. 2007, Varian 2007) and provides a convenient framework for researchers to explore the impact of budgets in position auctions. In practice, the position outcome could be random so that an advertiser with the highest bid is not ensured to win the highest position.\(^{15}\) For example, some large publishers (e.g., Google) rank ads based on the multiplication of an advertiser’s bid and a stochastic element. To capture this potential randomness in the position outcome, we revise the model from our main analysis by allowing the ad rank to be decided by advertisers’ bids weighted by a positive stochastic term \( \exp(\xi) \). Then an advertiser’s expected daily profit becomes,

\[
E \Pi_i(b_i, b_j) = \Pr \left( \frac{b_i \exp(\xi_i)}{b_j \exp(\xi_j)} \geq 1 \right) \Pi_1^i(b_j, B_i) + \Pr \left( \frac{b_i \exp(\xi_i)}{b_j \exp(\xi_j)} \leq 1 \right) \Pi_2^i(b_i, B_j),
\]

where \( \Pi_1^i(b_j, B_i) \) stands for the profit of staying at the first position and \( \Pi_2^i(b_i, B_j) \) represents the profit of staying at the second position. We further assume that \( \xi \) follows i.i.d. type 1 EV distribution, which implies that \( \Pr \left( \frac{b_i \exp(\xi_i)}{b_j \exp(\xi_j)} \geq 1 \right) = \Pr(\log(b_i) + \xi_i \geq \log(b_j) + \xi_j) = \frac{b_i}{b_i + b_j} \). To simplify the analysis, we assume the reservation value to be zero in this section.

**Proposition 7.** In a position auction with stochastic outcome, the publisher’s revenue strictly decreases when \( B_i \) exceeds the threshold \( B_i^* \) given any \( B_j \geq B_j^* \).

---

\(^{15}\) We thank an anonymous reviewer for this point.
We find that our main result regarding the non-monotonic relationship between the publisher’s revenue and advertisers’ budgets still holds. However, the intuition behind this result is quite different in that bid jamming will never occur at equilibrium as bidding one cent less is no longer a best-response bid in a stochastic setting.

We provide brief explanations for the result found in this extension. When both advertisers’ budgets $B_i$ and $B_j$ are sufficient, each advertiser’s bid only affects the probability distribution of staying between two positions but not the payoff at each position due to the second-price auction. Thus, an advertiser’s choice of bid is similar to the choice of a mixed strategy in a discrete game. Similar to the characterization for a mixed-strategy equilibrium, the equilibrium bids $(b_i, b_j)$ here are determined by conditions under which both advertisers are indifferent between two positions. Now imagine that $B_i$ declines below the sufficient-budget threshold so that advertiser $j$ has an incentive to raise the bid because of the jamming opportunity at the second position. To account for advertiser $j$’s incentive to raise the bid, advertiser $i$ in this case will increase the bid thereby escalating advertiser $j$’s cost at the first position. On the other hand, as advertiser $i$’s profit at the first position drops due to the decrease in budget, advertiser $j$ will slightly decrease her bid in equilibrium to keep advertiser $i$ indifferent between two positions. Thus, when $B_i$ declines below the sufficient-budget threshold, advertiser $i$’s bid will strictly increase (to maintain a “lean and hungry” look) while advertiser $j$’s bid is nearly unchanged. This change of equilibrium bids strictly benefits the publisher when advertiser $i$’s budget decreases.

G3. Proofs for Model Extensions

G3.1. Position Auctions with Incomplete Information

As discussed in Appendix G1., we assume (1) $B^H \geq B^*_i$; and (2) \[ \frac{(\pi_i - r)B^*_i}{\sigma \pi_i - B^*_j} \leq B^L \leq B^*_j, \] which is equivalent to $B^L \leq B^*_j \leq \hat{B}_i(B^L)$.

Lemma G1. One Bayes-Nash equilibrium of the position auction with incomplete information is: (1) When $\rho \leq \rho^*$, $b_i(B^L) = \hat{b}_i(B^L)$, $b_i(B^H) = b^*_i$, and $b_j(B^L) = b_j(B^H) = \frac{(\sigma - 1)\pi_j B^L}{\sigma^2 + \sigma \rho + (1 - \rho)}$; (2) When $\rho \geq \rho^*$, $b_i(B^L) = \hat{b}_i(B^L)$, $b_i(B^H) = b^*_i$, and $b_j(B^L) = b_j(B^H) = r$, where $\rho^* = \frac{\sigma (\sigma - 1)\pi_j - B^L}{\sigma (\sigma - 1)\pi_j - B^L}$.

Proof of Lemma G1. We verify that the equilibrium bids described above form a Bayes-Nash equilibrium. We start with the case where $\rho \leq \rho^*$, which is equivalent to $B^L \leq \sqrt{\frac{(\sigma - 1)\pi_j B^L}{\rho \sigma^2 + \sigma \rho + (1 - \rho)}}$. For advertiser $i$, we prove that she has no incentive to deviate from her current bid. When $B_i = B^L$, her highest expected profit of bidding below $b_j$ is $\Pi_i(b) = (1 - \rho)[(\pi_i - b) \frac{\sigma L}{\sigma b} + \sigma (\pi_i - b)(1 - \frac{B^L}{\sigma b})] + \rho (\pi_i - b)$, which is concave in $b$. Thus, the optimal bid $b_i = \frac{(\sigma - 1)\pi_i B^L}{\sigma^2 + \sigma \rho + (1 - \rho)}$ can be derived from FOC.

Because $\pi_i > \pi_j$, we have $\sqrt{\frac{(\sigma - 1)\pi_i B^L}{\sigma^2 + \sigma \rho + (1 - \rho)}} > \sqrt{\frac{(\sigma - 1)\pi_j B^L}{\sigma^2 + \sigma \rho + (1 - \rho)}} = b_j$, which indicates that advertiser $i$ cannot achieve this optimal bid if she decides to bid below $b_j$. Thus, her optimal bid below $b_j$ should be either
$b_j - \epsilon$ or $r$. If advertiser $i$ chooses $b_j - \epsilon$, then we have $\Pi_i(b_j - \epsilon) = (1 - \rho) \left( \frac{\pi_i B_j}{b_j} - B^L \right) + \rho(\pi_i - b_j) < \left( \frac{\pi_i B_L}{b_j} - B^L \right) = \Pi_i(\tilde{b}_i(B^L))$ because $b_j = \sqrt{\frac{(\sigma-1)\pi_i b_j}{\sigma^2 + \frac{\sigma b}{1-\rho}}} < \sqrt{\frac{(\sigma-1)\pi_i B^L}{\sigma^2}} = \tilde{b}_i(B^L)$. The last inequality is derived by Lemma C4, which shows $\tilde{b}_i(B^L) > \tilde{b}_i(B^L|r = 0) > \tilde{b}_j(B^L|r = 0) = \sqrt{\frac{(\sigma-1)\pi_i B^L}{\sigma^2}} > b_j$ when $B^L$ is in the range where $\tilde{b}_i > \tilde{b}_j$. Since we assume $B^L$ is large enough so that $\tilde{b}_i > \tilde{b}_j$ (where position paradox does not occur), we have proved $\Pi_i(\tilde{b}_i(B^L)) > \Pi_i(b_j - \epsilon)$. If advertiser $i$ chooses $r$, then we have $\Pi_i(r) = \pi_i - r < \left( \frac{\pi_i B_j}{b_j} - B^L \right) = \Pi_i(\tilde{b}_i(B^L))$ because $b_j < \tilde{b}_j(B^L|r = 0) < \tilde{b}_j(B^L)$. Thus, we have proved that $b_i(B^L) = \tilde{b}_i(B^L)$ is a best-response bid to $b_j$. Similarly, it is easy to check that $b_i(B^H) = b^*_j$ is also a best response to $b_j$. As for advertiser $j$, we argue that bidding $b_j = \sqrt{\frac{(\sigma-1)\pi_j B^L}{\sigma^2 + \frac{\sigma b}{1-\rho}}}$ is her best response to $b_i$ either when $B_j = B^L$ or when $B_j = B^H$. The reason is that $b_j$ is already a best-response bid if staying at the second position and advertiser $j$ has no incentive to bid over $b_j$ because $b_i \geq b^*_j$ under the assumption that $\frac{\pi_i - \pi_j}{\sigma b} \leq B^L$, which is equivalent to $\tilde{b}_i(B^L) \geq b^*_j$.

When $\rho$ increases above $\rho^*$, advertiser $j$’s optimal bid cannot be achieved because advertiser $i$ has no jamming apprehension given $b_j = \sqrt{\frac{(\sigma-1)\pi_j B^L}{\sigma^2 + \frac{\sigma b}{1-\rho}}}$. Due to the concavity of the optimal expected profit function at the second position, using bid jamming by bidding at $b_j = \tilde{b}_i(B^L) - \epsilon$ gives rise to less profit than $\Pi_j(b) = (1 - \rho)\left( [\pi_j - b] \frac{b^L}{\sigma b} + \sigma[\pi_j - b](1 - \frac{b^L}{\sigma b}) \right] + \rho(\pi_j - b) = (\pi_j - b)(1 - \rho)\left( \sigma - (\sigma - 1) \frac{b^L}{\sigma b} \right) + \rho < (\pi_j - b)(1 - \rho)\left( \sigma - (\sigma - 1) \right) + \rho = (\pi_j - b) < (\pi_j - r) = \Pi_j(r)$ because $\frac{b^L}{\sigma b} > 1$ when $\rho > \rho^*$, where $b = \sqrt{\frac{(\sigma-1)\pi_j B^L}{\sigma^2 + \frac{\sigma b}{1-\rho}}}$. Therefore, advertiser $j$’s optimal bid is $r$ independent of her budget. Meanwhile, advertiser $i$’s bid is still $b_i(B^L) = \tilde{b}_i(B^L)$ and $b_i(B^H) = b^*_j$. □

Proof of Proposition 6. We compute the publisher’s revenue based on Lemma G1. When $\rho \leq \rho^*$, $\Pi_P = (1 - \rho)\left( B^L + \frac{b B^L}{\sigma b} + \sigma b \left( 1 - \frac{B^L}{\sigma b} \right) \right) + \rho(\sigma + 1)b$, where $b = \frac{(\sigma - 1)\pi_j B^L}{\sigma^2 + \frac{\sigma b}{1-\rho}}$. $\Pi_P$ can be further simplified as $\Pi_P = (1 - \rho)\left( \sigma b + \frac{B^L}{\sigma} \right) + \rho(\sigma + 1)b$. It is easy to check that $\Pi_P'(\rho) = \left( b - \frac{B^L}{\sigma} \right) - \frac{\rho + \sigma b}{2\sigma(1-\rho^2)} \frac{2}{\sigma(1-\rho)} \left( 1 + \rho \frac{b}{\sigma(1-\rho)} \right)^{-\frac{1}{2}}$. Obviously, $\Pi_P'(\rho^*) < 0$ because $\sigma b(\rho^*) = B^L$. This already suggests that $\Pi_P'(\rho) < 0$ when $\rho$ approaches $\rho^*$ from the left. Furthermore, we have $\Pi_P = (\sigma + 1)r$ when $\rho > \rho^*$, which suggests that the publisher’s revenue strictly declines when $\rho$ exceeds $\rho^*$. Finally, as the expected budget $\rho B^H +$
(1 − ρ)B^L is a linear increasing function of ρ, we have proved that Π_i strictly decreases when the expected budget exceeds the threshold ρ^*B^H + (1 − ρ^*)B^L. □

G.3.2. Position Auctions with Stochastic Outcome

In this model extension, it is easy to verify that one advertiser with an extremely small bid and another advertiser with an extremely large bid is always an equilibrium. However, the advertiser with an extremely large bid always stays at the first position and therefore the publisher’s revenue is at the minimal level under such an equilibrium. Thus, we do not consider this trivial equilibrium in this game and focus on equilibrium bids that generate the highest revenue for the publisher. The notations used below all refer to the values when r = 0. For example, b_i^∗ = b_i^∗(r = 0), b_i = b_i(r = 0), and B_{ji} = B_{ji}(r = 0).

**Lemma G2.** When both advertisers’ budgets are sufficient, (b_i, b_j) = (b_i^∗, b_j^∗) at equilibrium. When advertiser j’s budget is sufficient, (b_i, b_j) = \left(\frac{2b_i^*B_{ji}^2}{B_i^2 - B_{ji}^2}, b_i^*\right) if B_i \in [B_{ji}, B_i^*) and only trivial equilibrium exists if B_i < B_{ji}. When advertiser i’s budget is sufficient and advertiser j’s budget is insufficient, only trivial equilibrium exists.

**Proof of Lemma G2.** We first prove that when both advertisers’ budgets are sufficient, the only nontrivial equilibrium bids are (b_i, b_j) = (b_i^∗, b_j^∗). When B_i ≥ B_i^∗ & B_j ≥ B_j^∗, advertiser i’s profit is

\[\Pi_i(b_i) = \frac{b_i[\sigma(\pi_i - b_i)]}{b_i + b_j} + \frac{b_j\pi_i}{b_i + b_j}\]

Its FOC is \[\Pi_i'(b_i) = \frac{\sigma b_j(b_i^*- b_j)}{(b_i + b_j)^2}\]. Thus, advertiser i is indifferent among any bids when b_j = b_i^∗. Similarly, advertiser j is indifferent among any bids when b_i = b_j^∗, which suggests that (b_i, b_j) = (b_i^∗, b_j^∗) is the only nontrivial equilibrium.

Next we consider the case in which B_j ≥ B_j^∗ but B_i < B_i^∗. Now advertiser i’s profit becomes

\[\Pi_i(b_i) = \frac{b_i}{b_i + b_j}\left[\frac{\pi_i B_i}{b_i} - B_i\right] + \frac{b_j\pi_i}{b_i + b_j}\] because advertiser i has jamming apprehension if she stays at the first position. It is easy to verify that advertiser i’s best-response is b_i(b_i) = \begin{cases} b_i, & \text{if } b_j > \hat{b_i} \\ \text{any } b, & \text{if } b_j = \hat{b_i}, \text{ where } b \\ \hat{b_i}, & \text{if } b_j < \hat{b_i} \end{cases}

and \hat{b} represent the lower and the upper bounds of the support of bids. As for advertiser j, Π_j(b_j) = \frac{b_j^2}{b_i + b_j}\left[\sigma(\pi_j - b_i)\right] + \frac{b_j\pi_j}{b_i + b_j}\left[\frac{\pi_j B_i}{b_i} + \sigma \pi_j \left(1 - \frac{B_i}{\sigma b_j}\right)\right]. It is easy to verify that Π_j'(b_j) = 0 is equivalent to b_j(b_j) = \frac{b_j^2 B_i}{\sigma b_i} + \frac{\left(\frac{\pi_j B_i}{\sigma b_i}\right)^2}{\pi_j} + \frac{b_j^2 B_i}{\sigma} , which can be also expressed as b_i(b_j) = \frac{2b_i b_j^2 B_i}{(\sigma b_2 - b_j^2) b_i}. Furthermore, Π_i''(b_i) < 0 can also be verified. Thus, the only nontrivial equilibrium bids are (b_i, b_j) = \left(\frac{2b_i B_{ji}^2}{B_i^2 - B_{ji}^2}, \hat{b_i}\right).

The last thing is to verify that advertiser i has jamming apprehension and advertiser j has jamming intention under this pair of bids. The former is true because \[\hat{b_i} < \sigma \hat{b_i} \text{ when } B_i < B_i^*\] and the latter is true because \[\frac{2\hat{b_i} B_{ji}^2}{B_i^2 - B_{ji}^2} > B_i \iff \frac{2\hat{b_i} b_j^2 B_i}{B_i^2 - B_{ji}^2} > \hat{b_i} \iff \left(2\hat{b_i} B_{ji}^2 - \hat{b_i}^2\right) + \hat{b_i}^2 B_i > 0, \] which is satisfied as long as the ratio between advertisers’ value-per-clicks is not too large. Because \[\frac{2b_i B_{ji}^2}{B_i^2 - B_{ji}^2} > 0\] is equivalent to \[\hat{b_i} > \hat{b_{ji}},\]
which is also equivalent to \( B_i > B_{ij} \) from Lemma C4, we have proved that only trivial equilibrium exists when \( B_i < B_{ij} \).

Finally, when \( B_i \geq B_i^* \) but \( B_j \leq B_j^* \), and Lemma C4 indicates that \( \hat{B}_j \leq \tilde{B}_{ij} \), the analysis above implies that only trivial equilibrium exists. \( \square \)

**Proof of Proposition 7.** When both advertisers’ budgets are sufficient, because \((b_i,b_j) = (b_i^*, b_j^*)\) at equilibrium, the publisher’s revenue is \( \Pi_p = \frac{b_j^* \sigma b_i^*}{b_j^* + b_i^*} + \frac{b_i^* \sigma b_j^*}{b_i^* + b_j^*} = \frac{2b_i^*b_j^*}{b_i^* + b_j^*} \). When \( B_i \in [B_{ij}, B_i^*] \) and \( B_j \geq B_j^* \), because \((b_i,b_j) = \left( \frac{2b_i^*b_j^*}{b_i^2 - B_{ij}}, \tilde{B}_i \right) \), we have \( \Pi_p = \frac{b_i^*b_j^*}{b_i^* + b_j^*} = \frac{b_i^*}{1 + \frac{b_i^*}{B_i^*}} = \frac{B_i + B_{ij}}{B_i^* + B_{ij}} = \frac{2b_i^*b_j^*}{B_i^* + B_j^*} = \frac{2b_i^*b_j^*}{B_i^* + B_j^*} \).

As the numerator increases with \( B_i \) while the denominator decreases with \( B_i \), we have shown that \( \frac{\partial \Pi_p}{\partial B_i} \geq 0 \) when \( B_i \in [B_{ij}, B_i^*] \). Furthermore, when \( B_i \to B_i^* \), \( \Pi_p \to \frac{4b_i^*b_j^*}{B_i^* + B_j^*} \), which indicates that \( \Pi_p \) decreases with \( B_i \) when \( B_i \) surpasses the sufficient budget threshold. Finally, \( \Pi_p = \sigma b_i \) when \( B_i < B_{ij} \). Thus, we have proved that the publisher’s revenue has an inverted-U relationship with \( B_i \) when \( B_j \) is sufficient.
References


